ES 111 Mathematical Methods in the Earth Sciences
Lecture Outline 16 - Tues 3rd Dec 2019

Constant Coefficient Nonhomogeneous LDE’s

So far we have dealt with \emph{homogeneous} linear ODE’s, for example

\[ a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0 \]  

(1)

This is homogeneous because the right-hand side is zero. But often we encounter ODE’s where the right-hand side (often called the “forcing function”) is non-zero. These are called \emph{nonhomogeneous} ODE’s; for example

\[ a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = F(x) \]  

(2)

Solving these kinds of equations is actually fairly straightforward, but it involves two steps.

The \textit{first step} is to solve the equivalent homogeneous ODE, which is called the \emph{complementary equation}. For example, equation (1) is the complementary equation of equation (2). We solve the complementary equation using the techniques given in Lecture 15.

The \textit{second step} is to use the extra information provided by the forcing function $F(x)$. There are various methods of doing so, but we’ll use the simplest, which is called the \emph{method of undetermined coefficients}.

This method assumes that the solution, called the \emph{particular solution}, has a similar form to the forcing function $F(x)$. For instance, if $F(x) = x^2$ we assume a polynomial of the form $y = Ax^2 + Bx + C$. If $F(x) = e^{ax}$ we assume a particular solution of the form $y = Ae^{ax}$. If $F(x)$ is $\sin ax$ or $\cos ax$, then we assume a particular solution of the form $y = A \cos ax + B \sin ax$. Note that in this case we are using both sine and cosine terms.

By substituting the assumed form of the particular solution back into equation (2), we can solve for the unknown constants $A, B, \cdots$.

The \emph{general solution} is then just the \emph{sum} of the solution to the complementary equation (first step) and the particular solution (second step). This solution will still have unknowns, which can be obtained as usual if the boundary conditions are known.
Example Solve the equation
\[
\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = x^2
\]

Answer The solution to the complementary equation is \(y = c_1e^x + c_2e^{-2x}\). We assume a particular solution of the form \(Ax^2 + Bx + C\). By substituting this into the ODE and rearranging we get 
\[-2Ax^2 + (2A - 2B)x + (2A + B - 2C) = x^2\]. This means that 
\[-2A = 1, 2A - 2B = 0\] and \(2A + B - 2C = 0\) which allows us to obtain \(A = -1/2, B = -1/2, C = -3/4\). So the general solution is
\[y = c_1e^x + c_2e^{-2x} - x^2/2 - x/2 - 3/4\]

Example Solve the equation
\[
\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = \sin x
\]

Answer The solution to the complementary equation is the same as before. For the particular solution, this time we assume \(y = A \cos x + B \sin x\). Substituting back in, we get 
\[-3A + B \cos x + (-A - 3B) \sin x = \sin x\], which allows us to solve for \(A\) and \(B\), giving \(A = -1/10\) and \(B = -3/10\). So the general solution is
\[y = c_1e^x + c_2e^{-2x} - \frac{1}{10}(\cos x + 3 \sin x)\]

If the forcing function is a product, then we try particular solutions which are products. For instance, if \(F(x) = x \cos x\) we would try a solution of form \((A + Bx) \cos x + (C + Dx) \sin x\).

Sometimes the most obvious particular solution is itself a solution to the complementary equation. In that case, multiply the original particular solution by \(x\) (or \(x^2\) if necessary).

There are other approaches to finding the particular solution, some of which include infinite series, but they are beyond the scope of this course.

**Simple Harmonic Oscillators**

A classic use of second-order ODE’s is to describe simple harmonic oscillators. Many physical systems fit into this general category, including \(L - C - R\) electric circuits, tidally-forced satellites, oscillations of the Earth etc.

A simple kind of oscillator is described by the homogeneous ODE
\[
m \frac{d^2x}{dt^2} + kx = 0
\]
where $x$ is displacement and $t$ is time. The quantity $m$ represents, for example, the mass on a spring, in which case $k$ is the spring constant, which describes how stiff the spring is. The solution to the ODE results in sinusoidal oscillations with an angular frequency $\omega = \sqrt{k/m}$ (how do you show this?). The frequency increases if the spring is stiffer (higher $k$) but decreases if the mass $m$ is larger. The oscillations persist indefinitely.

A slightly more complicated oscillator is one which is driven at a particular forcing frequency $\omega_0$. In this case the ODE is non-homogeneous:

$$m\frac{d^2x}{dt^2} + kx = F_0 \cos \omega_0 t$$

where $F_0$ is the amplitude of the forcing.

We can use the methods described above to solve for $x(t)$ in this case, and we obtain

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t + \frac{F_0}{m(\omega^2 - \omega_0^2)} \cos \omega_0 t$$

The system responds with two frequencies: the natural frequency $\omega$ and the forcing frequency $\omega_0$. Note that something very interesting happens if the forcing frequency ($\omega_0$) is the same as the natural frequency of the system ($\omega$): the amplitude of $x$ becomes infinite! This is the phenomenon known as resonance: if an oscillatory system is forced close to or at one of its natural frequencies, very large amplitude oscillations result. This is the reason, for example, that the Tacoma Narrows bridge failed.

In reality, of course, oscillations don’t become infinitely big. This is partly because any real system will experience damping, due to air resistance, eddy currents or whatever. We can modify our original ODE to include damping, as follows:

$$m\frac{d^2x}{dt^2} + 2c\frac{dx}{dt} + kx = 0$$

where here the damping is assumed to be proportional to velocity ($\frac{dx}{dt}$) and the factor of 2 is there for convenience. We can solve this HLDE as usual, and obtain

$$x(t) = c_1 e^{-\lambda t} \cos \sqrt{\omega^2 - \lambda^2} t + c_4 e^{-\lambda t} \sin \sqrt{\omega^2 - \lambda^2} t$$

where $\omega$ is the same as for the undamped system, we have defined $\lambda = c/m$ and have assumed that $c^2 < mk$ (this is a so-called underdamped system). Notice that the damping has two effects.
First, it changes the natural frequency of the system (which would be $\omega$ if there were no damping). Second, the amplitude decays with time at a rate controlled by $\lambda$. After a long time, the oscillations will have decayed to zero.

We can put all the pieces together and solve for the behaviour of a *forced, damped oscillator*. This is a bit tedious, but what you find is that after a long time the system responds at the forcing frequency, but with a *phase lag*. And the amplitude of the response is large if the forcing frequency is close to the natural frequency, but it doesn’t become infinite because of the effect of the damping.

**Reminders**

Problem Set 9 (optional) is posted.

The lecture on Thursday will be a revision lecture.