Integrating Factor

Here is a powerful technique which will work (only!) for linear first-order ordinary differential equations. If a first-order equation is not separable (see Lecture 14) then this technique is the next one to try.

Any such equation can be written in the so-called Standard Form

\[
\frac{dy}{dx} + p(x)y = q(x)
\]

The trick to solving this equation is to multiply both sides by an integrating factor \( w(x) \) where

\[ w(x) = e^{\int p(x)dx} \]

Doing so allows us to rewrite the Standard Form as

\[
\frac{d}{dx} \left[ ye^{\int p(x)dx} \right] = q(x)e^{\int p(x)dx}
\]

which we can always solve.

This looks more formidable than it really is (see example below). It’s a very powerful tool, but beware: it only works on linear first-order ODEs.

Example 1: Let’s solve our ooids problem from last time:

\[
\frac{dr}{dt} = a - br
\]

[Answer I.F.\(=\exp(bt)\) which yields \(re^{bt} = (a/b)e^{bt} + c\). At \(t = 0, r = 0\) so \(c = -(a/b)\). So the solution is \(r = (a/b)(1 - e^{-bt})\).]

Example 2: Now let’s solve a more complicated version of the ooid problem. Note that you can’t solve this example just by collecting all the \(r\)’s on one side and all the \(t\)’s on the other.

\[
\frac{dr}{dt} = ae^{-bt^2/2} - btr
\]

Does the answer make physical sense?
[Answer] I.F. = \exp(bt^2/2) which yields \( re^{bt^2/2} = at + c \). At \( t = 0, r = 0 \) so \( c = 0 \). So the solution is \( r = ate^{-bt^2/2} \) which means that as \( t \to \infty, r \to 0 \). This makes sense because the \(-btr\) term means the rate of shrinkage gets larger as time gets larger.]

**Example 3** Solve:

\[
2 \frac{dy}{dx} + 4xy = 2x
\]

[Answer] I.F. = \( \exp(x^2) \) which yields \( ye^{x^2} = e^{x^2} + c \). To solve for \( c \) the boundary conditions need to be specified.]

**Example** The deposition of sand grains in a fluid. Newton’s second law relates force \( F \) to the rate of change of momentum

\[
F = \frac{d}{dt}(mv) = m \frac{dv}{dt}
\]

where \( m, v \) and \( t \) are mass, velocity and time, respectively, and the second equality is obtained by assuming that mass (of a sand grain) is constant.

There are two forces acting on a particle settling out of suspension: the buoyancy force \( (B) \), which acts downwards, and the fluid drag force \( (D) \), which acts upwards. The buoyancy force is just \( \Delta mg \), where \( \Delta m \) is the excess mass of the particle and \( g \) is the acceleration due to gravity; note that this quantity is constant. The fluid drag force, however, is proportional to (and in the opposite direction to) the velocity of the falling particle:

\[
D = -kv
\]

So combining these equations, we get

\[
m \frac{dv}{dt} = \Delta mg - kv
\]

We can rewrite this in Standard Form:

\[
\frac{dv}{dt} + \frac{k}{m}v = \frac{\Delta mg}{m}
\]

and solve using the method outlined above. In this case the integrating factor \( w \) is given by

\[
w = e^{\frac{k}{m}t}
\]

and the general solution is

\[
v(t) = ce^{-kt/m} + \frac{\Delta mg}{k}
\]
If we assume that the particle was initially motionless \((v=0 \text{ at } t=0)\) then we can substitute in to our general solution and solve for \(c\): \(c = -\Delta mg/k\).

So the particular solution is:

\[
v(t) = \frac{\Delta mg}{k} \left(1 - e^{-kt/m}\right)
\]

As useful check on physical problems like this is to make sure that the units are consistent.

**What does this solution mean physically? What happens at short times (Maclaurin series) and long times?**

[**Answer** At short times, \(1 - e^{-kt/m} \approx (kt/m)\) by Maclaurin series expansion. So \(v(t) = gt\) which means that the particle is initially moving slowly so that it doesn’t experience any drag, it just accelerates due to gravity.]

If the drag force were not important, the velocity would increase continually due to the force of gravity. However, the drag force becomes more important and at some point the particle stops accelerating (when \(B\) and \(D\) cancel each other out). By setting \(t \to \infty\) we can find the asymptotic velocity

\[
v(\infty) \to \frac{\Delta mg}{k}
\]

This limiting velocity increases with increasing \(g\), and decreases with increasing drag \((k)\). Both of these effects are what you would expect based on physical intuition.

**Reality Checks** This example illustrates three reality checks that you can usually apply to make sure that your answer makes sense. They are:

1) Are the units consistent?

2) What happens in limiting cases (e.g. at short times, or \(t \to \infty\))? Do the results make sense?

3) What happens if you change a parameter value (e.g. increase \(g\))? Does it give the expected answer?

**Example** Fission track dating relies on the fact that the tracks are produced by uranium decay, but that they heal at a (temperature-dependent) rate. Let the amount of uranium be given by \(u(t) = u_0 \exp(-kt)\). Then the number of fission tracks \(f\) is given by

\[
\frac{df}{dt} = -\frac{du}{dt} - \lambda f
\]
where $\lambda$ gives the healing rate. How does the number of fission tracks vary with time? When does $f$ reach a maximum?

[Answer] The integrated form (I.F.) is $I.F. = \exp(\lambda t)$ which yields $fe^\lambda = \frac{ku_0}{\lambda-k} e^{(\lambda-k)t} + c$. If $f = 0$ at $t = 0$ then $c = -ku_0/(\lambda - k)$, so the solution is $f = \frac{ku_0}{\lambda-k} (e^{-kt} - e^{-\lambda t})$. The number initially climbs, then eventually decays once uranium is no longer making tracks. This answer gives the correct result when there is no healing ($\lambda = 0$) and always results in a positive number of fission tracks irrespective of whether $k$ or $\lambda$ is larger. At early times $f = ku_0 t$ so that the initial evolution is controlled by radioactive decay and not healing.]

Higher Order Differential Equations

Now we are going to move on to the higher-order differential equations (meaning derivatives higher than first order may appear). This is a big field, so we will focus mainly on solving specific, relatively simple higher order ODE’s which often crop up in Earth Sciences.

When are they useful? Especially in the equations of elasticity (e.g. seismic waves) and flexure. One kind of second-order DE describes simple harmonic motion.

The general form of a $n$-th order linear ODE is

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f(x)$$

where the $a_n(x)$ and $f(x)$ are functions of $x$. Written in this form $f(x)$ is known as the **input** or **driving** term (e.g. in elasticity, it represents the load being applied). A linear ODE with $f(x)=0$ is called **homogeneous**. Thus, a homogeneous linear ODE (HLDE for short) has no driving term. Here we will focus on HLDE’s.

Solutions to HLDE’s

In general, there is more than one solution to a HLDE. Suppose that $y_1(x)$ and $y_2(x)$ are both solutions of a HLDE, then any linear combination of them

$$c_1y_1(x) + c_2y_2(x)$$

is also a solution. This is true only of HLDE’s - if the driving term is not zero, then this ability to superpose solutions will not in general be true.

The general solution to an $n$-th order HLDE is constructed by a linear combination of $n$ **linearly independent** solutions to the HLDE. This is similar to the way we can construct any vector from a linear combination of the perpendicular vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$. 

A bunch of functions are linearly independent if and only if the only solution to
\[ c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) = 0 \]
is \( c_1 = c_2 = \cdots = c_n = 0 \) for some \( x \) in the range of interest. In other words, none of the functions can be constructed from a linear combination of the other functions.

A good way of thinking about this is to imagine graphs of the different functions and to see whether two graphs can be made to overlay each other at every point - if they can, then the functions are not linearly independent.

**Example** if \( y_1 = x \) and \( y_2 = x^2 \), are the two functions linearly independent? What about \( y_1 = x \) and \( y_2 = 3x \)? What about \( y_1 = \sin x \) and \( y_2 = \cos x \)?

If a bunch of simultaneous equations have a unique solution, then they must all be linearly independent, in which case their matrix determinant will be non-zero (see lecture 12). Finding out whether a bunch of functions are linearly independent can be done by constructing a matrix determinant called the Wronskian, which we will not deal with here.

**Particular Solutions of HLDE’s**

The particular solution of a HLDE is determined by specifying the free parameters (the \( c \)'s) in the general solution. For an \( n \)-th order HLDE, \( n \) initial conditions need to be specified to constrain the \( c \)'s.

**Obtaining Solutions to HLDE’s**

There are two methods of solving HLDE’s we’re going to address. The first is called *Reduction of Order* and is a bit of a cheat, since it assumes that you already know one solution to the equation, and is most useful with second-order HLDE’s. The advantage of the approach is that you end up solving a first-order ODE, which is much easier.

Let’s assume that the first (known) solution is \( y_1 \). How do we obtain this solution? Often, the best way is simply to guess! You can always substitute it in to see if it works. Given this solution, let the second (unknown) solution \( y_2 = v(x)y_1 \). By substituting this into the original HLDE and then substituting for \( w(x) = v'(x) \) you end up with a first-order ODE for \( w(x) \) which you can solve (using the *integrating factor* technique, or otherwise). Once this is solved, you obtain \( v(x) \) by integration and get the second solution because \( y_2 = v(x)y_1 \).
An example: find both solutions to

\[ y'' - 2y' + y = 0 \]

Another example: find the second solution to

\[ y'' - y = 0 \]

given that the first solution is \( e^x \).

[Answer Let the second solution be \( v(x)e^x \). Then we end up with \( v'' + 2v' = 0 \), so by integrating this gives \( v' = -2v \) which we can then solve to get \( v(x) = e^{-2x} + c \). So the second solution is \((e^{-2x} + c)e^x\) and the independent part of this solution is just \( e^{-x} \).]

Remember that the general solution is some linear combination of the individual solutions!

You can actually show that for a second-order HLDE the result is

\[ y_2(x) = y_1(x) \int \frac{1}{y_1^2(x)} e^{- \int \frac{a_1(x)}{a_2(x)} dx} dx \]

where the \( a \)’s are defined at the top of the outline. Although it’s not too hard to show where this comes from, I’m not going to show it to you: it’s much more useful to remember the general technique.

The next, more useful, method of solution is called the Constant Coefficients technique, because it works only for HLDE’s in which the \( a \)’s are not functions of \( x \), but constants. We’ll tackle these next lecture.