Differential equations (DE’s) are equations that contain derivatives of an unknown function, say \( y(x) \), that we seek to determine. DE’s turn up everywhere in geology: a simple example we will discuss below is radioactive decay. Whenever a physical law depends upon the rate of change (either temporal or spatial), then a DE is the result.

**Classes of DE’s**

A necessary preliminary is the nomenclature involved. There are different classes of DE’s:

1. **Ordinary/Partial**: A DE that has only one independent variable (e.g. \( x \)) is called ordinary, all others are partial. Ordinary DE’s have ordinary derivatives, partial DE’s have partial derivatives. In this class we will deal mainly with ordinary DE’s, but for many applications you need PDE’s (e.g. a cooling problem where temperature \( T \) is a function of both \( x \) and \( t \)).

2. **Order**: The order of a differential equation is the order of the highest derivative appearing in it (\( y', y'', \) etc.). The general form of a first-order ordinary DE is

\[
f(x, y, y') = 0 \quad \text{e.g.} \quad y' + e^x y = 0
\]

For a second-order ODE it is

\[
f(x, y, y', y'') = 0 \quad \text{e.g.} \quad y'' + ay' + e^x = 0
\]

and so on.

3. **Linearity**: A \( n \)th-order ordinary DE is linear if it can be written in the form

\[
a_0(x)y + a_1(x)y' + a_2(x)y'' + a_3(x)y''' + \cdots + a_n(x)y^{(n)} = f(x)
\]

where the \( a_i(x) \) and \( f(x) \) are functions of \( x \) only. So for example, the 1D equation of elastic flexure

\[
\frac{d^4y}{dx^4} + \alpha \frac{d^2y}{dx^2} + \beta y = f(x)
\]

is a linear, 4th-order ODE. An expression such as

\[
\left( \frac{dy}{dx} \right)^2 + y = f(x) \quad \text{or} \quad y'' y - y' = f(x)
\]
is non-linear and will not be treated in this course.

What does solving a first-order DE mean? We want to find a function $y = h(x)$ that satisfies our differential equation (here written as $f(x, y, y') = 0$) for all $x$ over the range of interest.

**Example** A typical first-order differential equation might be

$$\frac{dy}{dx} + 2y - 4 = 0$$

The solution to this is $y = 2 + ce^{-2x}$, where $c$ is an undetermined constant. You can see that this solution satisfies the differential equation by differentiating the solution and plugging in:

$$\frac{dy}{dx} + 2y - 4 = -2ce^{-2x} + 2(2 + ce^{-2x}) - 4 = 0$$

The *general solution* to a first order DE has infinitely many solutions which depend on a single free parameter (exactly like the undetermined constant when you integrate something). In general, an $n$th-order DE has a solution with $n$ free parameters. If you specify a particular value of the free parameter, you get a *particular solution*.

Particular solutions are obtained, i.e. the $n$ free parameters are determined, through satisfaction of $n$ initial (or boundary) conditions. These initial conditions are equations which specify the value of the function (e.g. $y(x)$) or some derivative of the function (e.g. $y'(x)$) at some specific value of $x$ (e.g. $x=0$). Boundary conditions are important!

Let’s take a very simple DE:

$$y' - f(x) = 0$$

which is linear ($a_1(x) = 1$) and first order. The solution is simply to integrate:

$$y = \int f(x) dx + c$$

This is the *general solution* with $c$ as the free parameter. To determine the value of $c$ and obtain a *particular solution* requires the specification of one initial condition e.g $y = 1$ at $x = 0$. Using this condition, we can then solve for $c$.

**Example** Solve the linear first-order DE

$$y' = x^2 + 1$$
By integrating the right-hand side, we obtain

\[ y = \frac{x^3}{3} + x + c \]

which is the *general solution*. If we also have the boundary condition that \( y = 1 \) at \( x = 0 \) then we solve for \( c \) by substituting these values into our general solution. This gives us \( c = 1 \), so the particular solution is simply

\[ y = \frac{x^3}{3} + x + 1 \]

An almost equally simple first order ODE has the form

\[ \frac{dy}{dx} = \frac{f(x)}{g(y)} \]

which can be written \( g(y)dy = f(x)dx \)

and is called *separable* since all the \( x \)-dependent terms can be collected on one side, and the \( y \)-dependent terms on the other. We can solve this equation very simply

\[ \int g(y)dy = \int f(x)dx + c \]

Note that here we have combined the two integration constants into a single constant \( c \).

An example of a separable ODE is radioactive decay:

\[ \frac{dN}{dt} = -kN \]

where \( N \) is the number of radioactive atoms left, \( t \) is time and \( k \) is the decay constant (with units of \( 1/\text{time} \)). We can separate this ODE

\[ \int \frac{dN}{N} = -\int kdt \]

and solve it (how?), obtaining

\[ N(t) = ce^{-kt} \]

If we also apply the initial condition \( N(0) = N_0 \), we obtain (how?) the particular solution

\[ N(t) = N_0e^{-kt} \]

which tells us that the radioactivity decays exponentially with time.

A *useful trick* to remember is that you can *always* check your solution to an ODE by differentiating and making sure you end back where you started.
Boundary Conditions are an important part of solving a differential equation. In general, an \( n \)-th order differential equation will need \( n \) boundary conditions to be specified. You use the boundary conditions to determine the values of the undetermined constants.

In the radioactive decay example above, our general solution is \( N = ce^{-kt} \) with \( c \) an undetermined constant. The boundary condition tells us that at \( t = 0 \), \( N = N_0 \). So we take our general solution and substitute in for \( t = 0 \), which gives \( N = c \). But our boundary condition is that \( N = N_0 \) at \( t = 0 \), which means that \( c = N_0 \). So the particular solution is \( N = N_0 e^{-kt} \).

Example The pressure \( P \) inside a uniform density planet is given by

\[
\frac{dP}{dz} = \frac{4}{3} \pi \rho^2 G (R - z)
\]

where \( G \) is the gravitational constant, \( \rho \) is the density, \( R \) is the planetary radius and \( z \) is depth below the surface. What is the pressure at the centre of the planet?

[Answer General solution: \( P(z) = \frac{(4/3)\pi G \rho^2 (Rz - z^2/2)}{3} + c \). At the surface \( z = 0 \) and \( P = 0 \) which implies \( c = 0 \). So the pressure at the centre \( (z = R) \) is given by \( P = \frac{(4/3)\pi G \rho^2 R^2 / 2}{3} \) or \( P = \rho g_0 R / 2 \) where \( g_0 \) is gravity at the surface.]

Another example How would you go about solving this equation?

\[
\frac{1}{x} \frac{dy}{dx} = -3(y + 1)
\]

[Answer Separable, giving \(-3x^2/2 + c = \ln(y + 1)\).]

A penultimate example The growth of oolites may be modelled in the following way:

\[
\frac{dr}{dt} = a - br
\]

where \( r \) is the oolite radius, \( a \) is the growth rate (constant, and determined by the background carbonate concentration) and \( b \) is an erosion rate which increases with increasing radius. How does the radius \( r \) change as a function of time? Does the answer make physical sense?

[Answer Separable, yielding \((-1/b) \ln(a-br) = t + c\). Taking \( r = 0 \) at \( t = 0 \) yields \( c = (-1/b) \ln(a) \). Substituting back in a rearranging gives \( r = (a/b)(1 - \exp(-bt)) \). At \( t = 0, r = 0 \) as required. For large \( t \), \( r \to (a/b) \) so that there is an equilibrium radius at which erosion and growth are in balance.]
**One more geological example** Materials which behave in an elastic fashion at high frequencies, and a viscous fashion at low frequencies (like silly putty) are called *viscoelastic*. For a viscoelastic material which is suddenly strained at $t = 0$ we may write its subsequent evolution as

$$\frac{d\sigma}{dt} = -\frac{E}{2\mu}\sigma$$

where $\sigma$ is stress, $E$ is Young’s modulus and $\mu$ is viscosity.

Solve this differential equation and explain why it makes physical sense. What is the characteristic relaxation timescale of the material?

[Answer] Separable, so $\ln \sigma = -(E/2\mu)t + c$. Taking $\sigma = \sigma_0$ at $t = 0$ we have $c = \ln \sigma_0$. Substituting and rearranging gives $\sigma = \sigma_0 \exp(-Et/2\mu)$. The stress decays due to viscous relaxation. The characteristic relaxation timescale is given by $2\mu/E$, which is known as the Maxwell time.]

**Reminders**

PS7 is due today 6pm.