Determinant of a Matrix

The determinant of a matrix is a scalar property of that matrix, which can be thought of physically as the volume enclosed by the row vectors of the matrix. Only square matrices have determinants. Determinants are also useful because they tell us whether or not a matrix can be inverted (see below).

The determinant of $A$ is written $|A|$ or $\det A$. Finding the determinant depends on the dimension of the matrix $A$: determinants only exist for square matrices. For a two by two matrix we have

$$\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Before extending this approach to larger matrices, let’s summarize some of the properties of this determinant:

1. If two rows of $A$ are equal (i.e. $a = c, b = d$) or if a row is expressible as a linear combination of other rows, then $\det A = 0$. The same is true for columns. If $\det A = 0$ then the matrix does not have an inverse (see below).

2. $\det A = \det A^T$

3. $\det(AB) = \det A \cdot \det B = \det(BA)$ for both $A$ and $B$ square.

4. If $A$ is upper or lower triangular, then $\det A = \Pi_{i=1}^n a_{ii}$

The determinant of a three by three matrix is

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Note the pattern of $+$ and $-$ signs.

The general formula is compact, but tedious to compute. Here it is for an $n$ by $n$ matrix $A$:

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$
where $C_{ij}$ are referred to as the **cofactors** and are computed from

$$C_{ij} = (-1)^{i+j} \det M_{ij}$$

The term $M_{ij}$ is known as the "minor matrix" and is the matrix you get if you eliminate row $i$ and column $j$ from matrix $A$. So to find the determinant of e.g. a $4 \times 4$ matrix, you end up calculating a bunch of $3 \times 3$ matrix determinants (much easier). Obviously you can apply this technique recursively (probably using a computer). Note how this general equation applies to the 3 by 3 matrix example given above.

Also note that another way of finding the determinant is to find its eigenvalues (see next lecture), though this technique is generally slower.

**What use are determinants?**

1. They tell you whether or not a matrix can be inverted (see below).
2. They are important in calculating eigenvalues and eigenvectors (see next lecture).
3. We use a matrix determinant to remember how to calculate a cross product or $\nabla \otimes$ (curl).
4. They are a measure of the area (or volume) of the shape defined by the rows of the matrix (treated as vectors). For instance
   $$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
defines a square in the $x - y$ plane. This shape has an area of 1, and its determinant also has a value of 1.

   You can show (**how?**) that a matrix
   $$A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$
has an area $(ad - bc)$.

If you end up with a negative determinant, that simply means that the shape is left-handed rather than right-handed.

**Solving a set of linear equations - Gaussian elimination**

Let’s return to our example problem

$$A \, x = b$$

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For a very simple system, this looks like

\[ a_{11}x_1 + a_{12}x_2 = b_1 \]
\[ a_{21}x_1 + a_{22}x_2 = b_2 \]

**How would we go about solving this?**

Gaussian elimination is just a way of formalizing this approach, using matrices. The procedure is to:

1) form the "augmented matrix"
2) reduce to "triangular form" by adding different combinations of rows together.
3) use "back-substitution" to obtain the unknowns (x’s).

It is easier to show an example of how this works in practice than describe it!

**Example**  The mineralogical example given in the last lecture.

Although Gaussian elimination if done by hand is very tedious, the technique works irrespective of the size of the matrices. Because of the power of modern computers, it is used in a great many numerical techniques, so it’s important to understand the principles.

A final caveat: Gaussian elimination is not guaranteed to work. You can end up with situations in which there is either no solution, or an infinite number of them. Whether a solution exists or not depends only on the matrix \( A \) and not the vector \( b \).

**Inverse Matrix**

If we have a square matrix \( A \) then it (usually) possesses an *inverse matrix* \( A^{-1} \) such that

\[ A^{-1} A = A A^{-1} = I \]

where \( I \) is the identity matrix. Inverse matrices obey the following rules:

1. \((AB)^{-1} = B^{-1} A^{-1}\)
2. \((A + B)^{-1} \neq A^{-1} + B^{-1}\)

The concept of an inverse matrix is important because it allows us to solve our usual linear equation

\[ A x = b \]

by pre-multiplying both sides (remember that for matrices the *order* in which we multiply things is important!):

\[ A^{-1} A x = A^{-1} b \]
So by obtaining $A^{-1}$ we can obtain $x$ directly. Thus, if $A^{-1}$ exists (which is not always the case), we are guaranteed a solution to the linear equation. How do we obtain $A^{-1}$? We do it by a variation on Gaussian elimination, called Gauss-Jordan elimination.

**Finding an Inverse**

First we have to check that an inverse actually exists. The quickest way to do this is to find $\det A$. If $\det A=0$ then there is no inverse, and the equation $Ax = b$ either has no solutions, or infinitely many. A matrix which cannot be inverted is called a *singular* matrix.

The most common reason for not being able to find an inverse is that one row (or column) is a linear combination of other rows or columns.

Assuming that $\det A \neq 0$ then we can find $A^{-1}$. In general, we do this by Gauss-Jordan elimination, which is tedious but works. The basic idea is to combine rows or columns of the original matrix (LHS) until it equals the identity matrix. At the same time, you carry out the same operations on the identity matrix (RHS). When the LHS is the identity matrix, the RHS will be the inverse matrix. The best way of learning it is by example:

**Example** Find the inverse (if it exists!) of

$$
\begin{pmatrix}
1 & 1 & 2 \\
2 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
$$

[Answer: The answer is:

$$
\begin{pmatrix}
-1 & 1 & 1 \\
2 & -1 & -3 \\
0 & 0 & 1
\end{pmatrix}
$$

but what is more important is the technique. I used $2r_1 - r_2 \rightarrow r_2$, $r_1 - 2r_3 \rightarrow r_1$, $r_2 - 3r_3 \rightarrow r_2$ and finally $r_1 - r_2 \rightarrow r_4$, but there are lots of alternatives.]

You can *always* check your answer by taking the product $AA^{-1}$ and making sure that the answer is the identity matrix $I$.

For a $2\times2$ matrix, the inverse is particularly simple. The inverse of

$$
A = \begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}
$$

is

$$
A^{-1} = \frac{1}{\det A} \begin{bmatrix}
a_{22} & -a_{12} \\
-a_{21} & a_{11}
\end{bmatrix}
$$
Check: does this work?

If you can find the inverse, you can find the solution to $Ax = b$. But if all you want is the solution to this equation, you don’t have to find the inverse: you can just do Gaussian elimination, which is simpler.

Reminder PS6 is due today at 6pm