Trigonometry

When is it useful? Everywhere! Anything involving coordinate systems (e.g. maps, orbits), resolving of forces, angular variations (e.g. statistics), etc. etc.

Given the triangle

we define

\[
\sin \theta = \frac{\text{opp.}}{\text{hyp.}} = \frac{o}{h}
\]

\[
\cos \theta = \frac{\text{adj.}}{\text{hyp.}} = \frac{a}{h}
\]

\[
\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\text{opp.}}{\text{adj.}} = \frac{o}{a}
\]

What do they look like?

Note that \( \cos(\theta) = \cos(-\theta) \) ("even" function) and \( \sin(\theta) = -\sin(-\theta) \) ("odd" function).

In 2D, Pythagoras’ theorem tells us that

\[
h^2 = o^2 + a^2
\]

Dividing by the square of the hypotenuse \( h^2 \) we get

\[
1 = \left( \frac{a}{h} \right)^2 + \left( \frac{o}{h} \right)^2 = \cos^2 \theta + \sin^2 \theta
\]

Note that \( \cos^2 \theta \) is the same thing as \( (\cos \theta)^2 \)
The angle $\theta$ is expressed in radians (which are unitless). When working in radians, it is simple to calculate the arc length $l$ subtended by the angle $\theta$:

$$l = r\theta$$

where $r$ is the radius of the circle.

![Figure 2](image)

Since the perimeter of a circle is $2\pi r$, then “$360^\circ = 2\pi$ radians. You should always use radians when doing math problems; degrees are for maps.

**Useful approximations** When $\theta$ is small ($\theta \approx 0$) we have

$$\sin \theta \approx \theta$$

$$\cos \theta \approx 1 - \frac{\theta^2}{2} \approx 1$$

These formulae only work if $\theta$ is in radians. We’ll see where they come from in Lecture 2.

Useful formulas include:

1) Double angle formulae (*where does this come from?*):

$$\sin(\theta + \phi) = \sin \theta \cos \phi + \cos \theta \sin \phi$$

$$\cos(\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi$$

You can also obtain $\sin$ and $\cos$ of $(\theta - \phi)$ by replacing $\phi$ with $-\phi$ in the above formulae. You can also obtain $\sin$ and $\cos$ of $2\theta$ by setting $\theta = \phi$.

For the following general triangle we also have:
2) Law of Cosines

\[ a^2 = b^2 + c^2 - 2bc \cos \alpha \]

3) Law of Sines

\[ \frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma} \]

**Example** apparent dip of a fault

**Another example:** roadcut

**Differentiation**

**When is it useful?** Anywhere that we are interested in changing quantities (fluid flux, topographic gradients, stock markets etc. etc.); also for finding maxima/minima.

We will denote any function of \( x \), for instance \( \sin(x) \), \( x^2 \), etc., as \( f(x) \). It is important to understand that if \( f(x) = x^2 \), then \( f(x + h) = (x + h)^2 \). Likewise, if \( f(x) = \cos(x) \), then \( f(x + h) = \cos(x + h) \) and so on.

When we take the slope of a line (i.e. its gradient) between two points \( (x_1, y_1) \) and \( (x_2, y_2) \) we calculate it by using

\[ \frac{y_2 - y_1}{x_2 - x_1} \]

If the \( y \) values are described by some function of \( x \), \( y = f(x) \), then we can rewrite the slope of the line as

\[ \frac{dy}{dx} = \frac{df}{dx} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \]

Incidentally, this makes it obvious that \( \frac{dx}{dy} = 1 / \frac{dy}{dx} \).
Now let’s instead write \( x_2 = x_1 + h \). Then as \( h \) becomes vanishingly small, we are calculating the slope of the line at position \( x_1 \). The slope of the line described by \( y = f(x) \) at position \( x \) is known as the derivative of \( f(x) \) with respect to \( x \) and is written \( \frac{df}{dx} \) or \( f'(x) \). Formally, we write

\[
\frac{df}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

**What does this mean?** Example: \( \sin x \)

The basic rules of differentiation are:

1. If \( f(x) = \text{constant} \) then \( \frac{df}{dx} = 0 \)
2. If \( f(x) = x^n \) then \( \frac{df}{dx} = nx^{n-1} \)
3. If \( F(x) = f(x) + g(x) \) then \( \frac{dF}{dx} = \frac{df}{dx} + \frac{dg}{dx} \)
4. If \( f(x) = u(x)v(x) \) then \( \frac{df}{dx} = u(x)\frac{dv}{dx} + v(x)\frac{du}{dx} \) (product rule)
5. If \( f(x) = g(u(x)) \) then \( \frac{df}{dx} = \frac{dg}{du} \cdot \frac{du}{dx} \) (chain rule)

Special functions that you’ll need to know are

- if \( f(x) = \ln(x) \) then \( \frac{df}{dx} = x^{-1} \)
- if \( f(x) = e^{ax} \) then \( \frac{df}{dx} = ae^{ax} \)
- if \( f(x) = \sin(x) \) then \( \frac{df}{dx} = \cos(x) \)
- if \( f(x) = \cos(x) \) then \( \frac{df}{dx} = -\sin(x) \)

To find the maximum/minimum of a function we find the value(s) of \( x \) at which

\[
\frac{df}{dx} = 0
\]

To determine whether these points are maxima or minima, we differentiate a second time:

- if \( \frac{d^2f}{dx^2} = f''(x) > 0 \) then minimum
- if \( \frac{d^2f}{dx^2} = f''(x) < 0 \) then maximum

**Why do things work this way? What happens if \( \frac{d^2f}{dx^2} = 0? \)** We’ll discuss these issues further in Lecture 8.

**Example: efficient box dimensions**
**Integration**

**Where is it useful?** Gravity, moment of inertia, energy . . . see examples below.

You can think of integration as the reverse of differentiation. Physically, differentiation involves taking the slope of a line, while integration involves totalling up the area under a curve.

Differentiation involves loss of information e.g. differentiating $f(x)$ and $f(x)+c$ will give the same answer ($f'(x)$). So when we integrate, we need to add in an undetermined constant of integration:

$$\int f'(x) dx = f(x) + c$$

where $c$ is a constant. This is an *indefinite integral* because the limits of integration are not defined. If they are defined, then instead we get

$$\int_a^b f'(x) dx = f(b) - f(a)$$

**Simple integration examples**

1. If $f(x) = x^n$ then $\int f(x) dx = \frac{x^{n+1}}{n+1} + c$
2. If $f(x) = \cos(2x)$ then $\int f(x) dx = \frac{1}{2} \sin(2x) + c$
3. If $f(x) = e^{ax}$ then $\int f(x) dx = \frac{1}{a} e^{ax} + c$
4. If $f(x) = \frac{1}{ax}$ then $\int f(x) dx = \frac{1}{a} \ln x + c$

Some examples of when integration is useful:

1. Area beneath a curve $f(x)$ between $x = a$ and $x = b$ is

   $$\text{Area} = \int_a^b f(x) dx$$

   Note that this expression has the correct units for area.

2. The average (mean) value of $f(x)$ between $x = a$ and $x = b$ is

   $$\text{Mean} = \frac{1}{b-a} \int_a^b f(x) dx$$

   Again, note that this expression has the correct units.

3. Recall that work = force times distance, or $dW = f(x) dx$ where $f(x)$ is the force, $dx$ is the increment in distance and $dW$ is the increment in work. The work done in moving from $a$ to $b$ is

   $$W = \int_a^b f(x) dx$$
Are the units correct? Example: meteor hitting the Earth

4. The last thing to remember is integration by parts. It is easiest to demonstrate how this works with an example: \( \int xe^{ax} dx \).

To show where this comes from, we start with the chain rule

\[
d(uv) = u \, dv + v \, du
\]

Integrating both sides yields

\[
\int d(uv) = \int u \, dv + \int v \, du
\]

or

\[
\int u \, dv = uv - \int v \, du
\]

Always choose \( u \) so that if you differentiate it enough times it will become a constant. Remember that you can always check your answer by differentiating!

Second example: \( e^x \sin x \)