ES 111 - Final 2016

Answer Question 1 and four (4) of Questions 2-9.
You have three (3) hours to complete the test.
No calculators or cheat sheets to be used (except the Formula Sheet I’ve provided you with).
Make sure you show all your working!

Question 1 [Everyone must answer this question]

a) For the vector field \( \mathbf{f} = [x^2 + y^2, 2xy, x^2 + z^2] \), find \( \nabla \otimes \mathbf{f} \). [1]

b) Find the equation of the plane passing through the points (1,1,1), (2,1,0) and (0,2,3). [2]

c) Find the following integral: \( \int \sin^2 x \, dx \). [2]

d) Solve the following first order ODE [2]:

\[
\frac{dy}{dx} + 2y = k
\]

\[ \text{Hint: Start with } (AB)(AB)^{-1} \] [2]

e) Using the Maclaurin series, or otherwise, find the first two non-zero terms in the series expansion of \( \cos^2 x \). [2]

f) Prove that \( (AB)^{-1} = B^{-1}A^{-1} \). (Hint: Start with \( AB(AB)^{-1} \)). [2]

g) Find the eigenvalues and eigenvectors of the following matrix [3]:

\[
\begin{pmatrix}
a & b \\ 
\frac{1}{b} & a
\end{pmatrix}
\]

h) Solve the following non-homogeneous ODE [4]:

\[
\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = e^x
\]

[18 total]

Answer four of questions 2-9

Question 2

a) You are given the equations of two lines which intersect each other:

\[(x - 4) = 3y = 3(z - 1)\]

and

\[2(x - 1) = y + 1 = z\]

For each line, write down a vector parallel to the line. [2]

b) Also write down the coordinates of the point where the two lines intersect (Hint: solve two simultaneous equations for \( x \) and \( y \) first) [3].

c) For each line, calculate the \( y \) and \( z \) coordinates when \( x = 0 \) [1].

d) Using your answers to a) and b), find the equation of a plane (in terms of \( x, y \) and \( z \)) which contains both of the lines. [4] [10 total]
**Question 3**

a) Find an expression for the eigenvalues $\lambda$ of the following matrix $\begin{pmatrix} a & c \\ c & b \end{pmatrix}$.

b) Show that $\lambda_1 + \lambda_2$ gives the trace of the matrix. [1]

c) Show that $\lambda_1 \lambda_2$ gives the determinant of the matrix. [2]

d) Find an expression for one eigenvector of this matrix, in terms of $\lambda$, $c$ and $a$. (Hint: set the $y$-component of the vector to 1). [1]

e) Take the dot product of the two eigenvectors. By using your answers to b) and c), or otherwise, show that the two eigenvectors are perpendicular. [3] [10 total]

**Question 4**

a) Using the expressions for $a \cdot b$ and $|a \otimes b|$ given in the formula sheet, show that $|a \otimes b|^2 = |a|^2|b|^2 - (a \cdot b)^2$. [3]

b) Given two vectors $a = [a_1, a_2, a_3]$ and $b = [b_1, b_2, b_3]$ which are parallel, write down expressions for $b_1, b_2$ and $b_3$ in terms of $a_1, a_2, a_3$ and a constant $c$. [1]

c) Hence use the definition of the cross product to show that the cross product of two parallel vectors is the null vector. [2]

d) Use the diagram on the board and the cosine rule to prove that $a \cdot b = |a||b| \cos \theta$. (Hint: using $a = [a_1, a_2]$ and $b = [b_1, b_2]$ will help). [4] [10 total]

**Question 5**

a) The wave equation is given by \[ \frac{\partial^2 s}{\partial t^2} = c^2 \frac{\partial^2 s}{\partial x^2} \] where $s$ is the displacement and $c$ is a velocity. Using separation of variables and taking $s = 0$ at $x=0$ and $s = 0$ at $t = 0$, show that one possible solution can include a term $\sin(\omega t)$. Here $\omega$ is just a constant. [5]

b) A sinusoidal wave having the form $\sin(\frac{2\pi}{\lambda} x)$ has a wavelength $\lambda$. Using your result to part a), write down the relationship between $\omega$, $c$ and $\lambda$. [1]

c) We’ll now add an extra term to equation (1):
\[ \frac{\partial^2 s}{\partial t^2} + 2p \frac{\partial s}{\partial t} = c^2 \frac{\partial^2 s}{\partial x^2} \]

Use separation of variables again to find the new solution just for the part that depends on $t$. You can assume that $p^2 - \omega^2 < 0$. [3]

d) Sketch a curve showing what the part that depends on $t$ looks like. [1] [10 total] [10 total]
Question 6

The change in pressure with elevation is given by

\[ \frac{dP}{dz} = \frac{-k}{T_0 - \beta z} P \]

where \( P \) is pressure, \( z \) is height, \( T_0 \) is the surface temperature (at \( z = 0 \)), \( \beta \) is the rate at which the atmosphere cools with increasing height, and \( k, \beta \) and \( T_0 \) are all constant.

a) Find the general solution to this equation. [4]

b) Find the particular solution, given that \( P = P_0 \) at \( z = 0 \). [1]

c) If \( \beta \) is small, find an approximation to your solution (the answer should be independent of \( \beta \)). [2]

d) The total column mass in the atmosphere is related to the quantity \( \int P(z)dz \). Find the following integral [3] [10 total]:

\[ \int_0^{T_0/\beta} P(z)dz \]

Question 7

a) For the following function:

\[ z = 8x^2 - 8xy + \frac{y^3}{3} + 3y \] (2)

find the \((x, y, z)\) coordinates of the critical points and identify their nature. [4]

b) Find the equation of the 3D line passing through these two critical points [3]

c) What is the \(z\)-value of this line at \(x = 0, y = 0\)? [1]

d) For the function described by equation (2), what direction is the steepest slope pointing in at \( x=0, y=0 \), and what is the slope? [2] [10 total]

Question 8

A toy model for the oscillations of the Earth is given by

\[ \frac{d^2y}{dt^2} + \frac{g}{R}y = F \sin \omega t \]

where \( y \) is the displacement, \( t \) is time, \( g \) is surface gravity, \( R \) is the radius of the Earth and the oscillation is forced at an amplitude \( F \) and angular frequency \( \omega \).

a) Find the complete solution to this non-homogeneous equation (you’ll have two undetermined constants, and you may assume that \( \omega \) does not equal \( \sqrt{g/R} \)). [6]

b) If the forcing frequency \( \omega \) does equal \( \sqrt{g/R} \), what happens to the amplitude of \( y \) in your solution to a)? What does \( \sqrt{g/R} \) represent? [2]

c) If the forcing frequency is less than \( \sqrt{g/R} \) by a small amount \( \delta \), find an approximate expression for the quantity

\[ \frac{F}{(g/R) - \omega^2} \]

[2] [10 total]
Question 9

a) Express $\cos x$ as a Maclaurin series up to and including a term in $x^2$. Show your working. [2]
b) Express $\sin x$ as a Maclaurin series up to and including a term in $x^3$. [2]
c) Express $e^{ix}$ as a Maclaurin series up to and including a term in $x^3$. Here $i = \sqrt{-1}$. [3]
d) Hence show that $e^{ix} = \cos x + i \sin x$ [1]
e) Also prove that $(\cos x + i \sin x)^n = \cos nx + i \sin nx$. [2] [10 total]