Note that this is \textit{not} a comprehensive list. There are things that don’t appear on this sheet that I will expect you to know (such as how \sin, \cos etc. are defined, or what the differential of $e^x$ is). You will also need to be able to understand and manipulate these expressions.

\textbf{Basic Trigonometry}

$$\cos^2 x + \sin^2 x = 1$$

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

cosine formula:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

sine formula:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

When $x \ll 1$:

$$\sin x \approx x \quad \text{and} \quad \cos x \approx 1 - \frac{x^2}{2}$$

\textbf{Basic Calculus}

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

product rule:

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$$

integration by parts:

$$\int u dv = uv - \int v du$$

Maclaurin series expansion:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$$

Useful examples:

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots \quad (1 + x)^n = 1 + nx + \cdots$$

\textbf{Vectors}

Vector $\mathbf{a} = [a_1, a_2, a_3]$

Unit vector:

$$\hat{a} = \frac{a}{|a|}, \quad |\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Dot product:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta = a_1b_1 + a_2b_2 + a_3b_3$$
Cross product:
\[ a \otimes b = [a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1] \]
\[ |a \otimes b| = |a||b| \sin \theta \]

**Lines and Planes**
Vector and algebraic equations of a line passing through \( r_0 = (x_0, y_0, z_0) \) parallel to \( \mathbf{v} = [a, b, c] \):
\[
 r = r_0 + tv, \quad \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}
\]

Vector and algebraic equations of a plane passing through \( r_0 = (x_0, y_0, z_0) \) perpendicular to \( \mathbf{n} = [a, b, c] \):
\[
 \mathbf{n} \cdot (r - r_0) = 0, \quad a(x - x_0) + b(y - y_0) + c(z - z_0) = 0
\]

**Partial Differentials**
\[
 \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}
\]
For \( z = f(x, y) \) the total differential \( dz \) is given by
\[
 dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy
\]
The gradient of \( z = f(x, y) \) in two dimensions is given by
\[
 \nabla f = \begin{bmatrix} \frac{\partial f}{\partial x}, & \frac{\partial f}{\partial y} \end{bmatrix}
\]
The directional derivative \( D_u f \) of \( f \) in the direction \( \hat{u} \) is given by
\[
 D_u f = \hat{u} \cdot \nabla f
\]
A critical point occurs when
\[
 \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0
\]
For a critical point at \((a, b)\)
\[
 D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2
\]
If \( D > 0 \) and \( f_{xx} \) (or \( f_{yy} \)) \( > 0 \): \( f(a, b) \) is a minimum
If \( D > 0 \) and \( f_{xx} \) (or \( f_{yy} \)) \( < 0 \): \( f(a, b) \) is a maximum
If \( D < 0 \): \( f(a, b) \) is not a local extremum (it’s a saddle point)
If \( D = 0 \): indeterminate

**Vector calculus**
The del operator \( \nabla = \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] \).
Gradient: \( \text{grad} f = \nabla f = \left[ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right] \) where \( f \) is a scalar field.
Divergence: \( \text{div} \mathbf{v} = \nabla \cdot \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \) where \( \mathbf{v} = [v_1, v_2, v_3] \) is a vector field.
Curl: \( \text{curl} \mathbf{v} = \nabla \otimes \mathbf{v} = \left[ \frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}, \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}, \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right] \) where \( \mathbf{v} = [v_1, v_2, v_3] \) is a vector field.
Laplacian: \( \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \) where \( f \) is a scalar field.
Matrices

Matrices $A$, and the components of a matrix $a_{ij}$, are always described with row first, then column.

Matrix multiplication $C = AB$

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

where $A$ has $n$ columns and $B$ has $n$ rows.

$$AB \neq BA \quad \text{(in general)}$$

$$(AB)^T = B^T A^T$$

where $^T$ denotes a transpose matrix.

An inverse matrix $A^{-1}$ is defined such that $A A^{-1} = A^{-1}A = I$, where $I$ is the identity matrix.

For a $2\times2$ matrix if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Here the term $(ad - bc)$ is the determinant of $A$, that is $|A|$. If $|A| = 0$, the matrix has no inverse.

$$|A B| = |A||B| = |B A| \quad \text{and also} \quad |A| = |A^T|$$

Gaussian elimination: make sure you know how to do it!

Eigenvectors/eigenvalues:

$$A x = \lambda x \quad \text{and therefore} \quad |A - \lambda I| = 0$$

An $n \times n$ matrix always has at most $n$ independent eigenvectors.

Ordinary Differential Equations

If a first-order equation is of the kind

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

then it is separable and may be solved by writing

$$\int f(x)dx = \int g(y)dy$$

If a first-order equation is in standard form:

$$y' + p(x)y = q(x)$$

then this equation can be solved by separation after multiplying both sides by the integrating factor:

$$e\int p(x)dx$$

Reduction of order: if one solution $(y_1)$ to a higher-order DE is known, a second solution may be found by writing $y_2 = v(x)y_1$ and substituting into the original equation.

Constant coefficients: if presented with a homogeneous DE of the form

$$a_n y^{(n)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0$$
where \(a_0, a_1, \ldots, a_n\) are constant, then there are \(n\) independent solutions each of the form

\[ y = c_ne^{m_nx}, \]

where \(c_n\) is an undetermined constant and \(m_n\) is a root of the equation

\[ a_nm^n + \cdots + a_2m^2 + a_1m + a_0 = 0 \]

These independent solutions may be combined into a single general solution

\[ y = c_1e^{m_1x} + c_2e^{m_2x} + \cdots + c_ne^{m_nx} \]

If you end up with repeated roots, then the second, third etc. independent solutions are \(xe^{m_nx}, x^2e^{m_nx}, \ldots\), where \(m\) is the root which is repeated.

If you end up with two complex roots (that is, \(m_1 = a + ib\) and \(m_2 = a - ib\)) then you can write the solution as

\[ y = c_1e^{ax}\cos bx + c_2e^{ax}\sin bx \]

If the equation is non-homogeneous with a right-hand side \(F(x)\) then you first solve the complementary equation with \(F(x) = 0\) and then find the particular solution by assuming a particular form based on \(F(x)\):

- If \(F(x)\) is a polynomial, then assume a solution of the form: \(A + Bx + Cx^2 + \cdots\).
- If \(F(x)\) is \(e^{kx}\), then assume a solution of the form: \(Ae^{kx}\).
- If \(F(x)\) is \(\sin x\) or \(\cos x\), then assume a solution of the form: \(A\cos x + B\sin x\).

The complete solution is the sum of the particular solution and the solution to the complementary equation.

**Higher Order Partial Differentials**

If you have a higher-order partial differential equation with a function that depends on two variables (e.g. \(T(x, t)\)), try writing \(T = A(x)B(t)\) which will give you two separate differential equations, one involving \(x\) only and one involving \(t\) only. This is called *Separation of Variables*.

The Laplacian of a function \(f(x, y, \cdots)\) in Cartesian coordinates is

\[ \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \cdots \]