ES 111 Mathematical Methods in the Earth Sciences

Equations that you should know

Note that this is not a comprehensive list. There are things that don’t appear on this sheet that I will expect you to know (such as how sin, cos etc. are defined, or what the differential of $e^x$ is). You will also need to be able to understand and manipulate these expressions.

**Basic Trigonometry**

\[
\cos^2 x + \sin^2 x = 1
\]

\[
\sin(x + y) = \sin x \cos y + \cos x \sin y
\]

\[
\cos(x + y) = \cos x \cos y - \sin x \sin y
\]

cosine formula:

\[
a^2 = b^2 + c^2 - 2bc \cos A
\]

sine formula:

\[
\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}
\]

When $x \ll 1$:

\[
\sin x \approx x \quad \text{and} \quad \cos x \approx 1 - \frac{x^2}{2}
\]

**Basic Calculus**

\[
\frac{df}{dx} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

product rule:

\[
\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}
\]

integration by parts:

\[
\int u dv = uv - \int v du
\]

Maclaurin series expansion:

\[
f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots
\]

**Vectors**

Vector $a = [a_1, a_2, a_3]$

Unit vector:

\[
\hat{a} = \frac{a}{|a|}, \quad |a| = \sqrt{a_1^2 + a_2^2 + a_3^2}
\]

Dot product:

\[
a \cdot b = |a||b| \cos \theta = a_1 b_1 + a_2 b_2 + a_3 b_3
\]

Cross product:

\[
a \otimes b = [a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1]
\]

\[
|a \otimes b| = |a||b| \sin \theta
\]
Lines and Planes
Vector and algebraic equations of a line passing through \( r_0 = (x_0, y_0, z_0) \) parallel to \( \mathbf{v} = [a, b, c] \):

\[
\mathbf{r} = r_0 + t\mathbf{v}, \quad \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}
\]

Vector and algebraic equations of a plane passing through \( r_0 = (x_0, y_0, z_0) \) perpendicular to \( \mathbf{n} = [a, b, c] \):

\[
\mathbf{n} \cdot (\mathbf{r} - r_0) = 0, \quad a(x-x_0) + b(y-y_0) + c(z-z_0) = 0
\]

Partial Differentials

\[
\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}
\]

For \( z = f(x, y) \) the total differential \( dz \) is given by

\[
dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy
\]

The gradient of \( z = f(x, y) \) in two dimensions is given by

\[
\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \end{bmatrix}
\]

The directional derivative \( D_u f \) of \( f \) in the direction \( \mathbf{u} \) is given by

\[
D_u f = \mathbf{u} \cdot \nabla f
\]

A critical point occurs when

\[
\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0
\]

For a critical point at \((a, b)\)

\[
D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2
\]

If \( D > 0 \) and \( f_{xx} \) (or \( f_{yy} \)) > 0: \( f(a, b) \) is a minimum
If \( D > 0 \) and \( f_{xx} \) (or \( f_{yy} \)) < 0: \( f(a, b) \) is a maximum
If \( D < 0 \): \( f(a, b) \) is not a local extremum (it’s a saddle point)
If \( D = 0 \): indeterminate

Vector calculus
The del operator \( \nabla = \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] \).

Gradient: \( \text{grad } f = \nabla f = \left[ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right] \) where \( f \) is a scalar field.

Divergence: \( \text{div } \mathbf{v} = \nabla \cdot \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \) where \( \mathbf{v} = [v_1, v_2, v_3] \) is a vector field.

Curl: \( \text{curl } \mathbf{v} = \nabla \times \mathbf{v} = \left[ \frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}, \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}, \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right] \) where \( \mathbf{v} = [v_1, v_2, v_3] \) is a vector field.

Laplacian: \( \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \) where \( f \) is a scalar field.