Shear-weakening of the transitional regime for granular flow

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This paper experimentally investigates the rheology of dense granular flow through its solid-like to fluid-like transition. Between the well-established flow regimes–quasi-static and grain-inertial–the physical description of the transition remains elusive. Our experiment utilizes a top-rotating torsional shear cell capable of \( \pm 1 \mu m \) accuracy in height and 5 decades \((10^{-3} - 100 \text{ rad s}^{-1})\) in rotation rate. The data on beach sand shows that shear and normal stresses exhibit an inverse rate-dependence under a controlled volume environment in the transitional regime, while in the limiting regimes the results are in agreement with previous work. The shear-weakening stresses illustrates a previously unknown ‘dip’ with increasing shear-rate. Under a controlled pressure environment, however, the shear-compacting volume-fraction ‘peaks’ with increasing shear-rate. We combine these results from both configurations to infer a constitutive law based on a rate-invariant granular fluid compressibility. The formulation provides an equation-of-state for dynamic granular systems, with state variables of pressure, strain-rate, and free-volume-fraction. Fitting parameters from independent constant-volume and constant-pressure data shows good agreement in validating our model. Moreover, the degree of grain jaggedness is essential to the rate-dependence within the transitional regime. The results on the solid-fluid transition may elucidate the evolution of granular flow anisotropies.

1. Introduction

Under continuous strain, granular material display regimes analogous to all three phases of matter, i.e. solid, liquid, and gas (see Jaeger, Nagel, & Behringer 1996). These phases have distinct modes of momentum transfer between grains under different amounts of external excitations–surface shear, air injection, or vibration. With small excitations and little or no confinement, granular material escapes from a jammed state into a mobilized one (see Cates, Wittmer et al. 1999; O’Hern, Langer et al. 2001; Corwin, Jaeger, & Nagel 2005). Within this regime, the long-duration grain-to-grain interaction is frictional Coulomb-type with momentum exchanging mainly through rubbing and rolling. The interactions often involve many grains in aggregated forms (see Savage, Nedderman et al. 1983; Savage 1984) within this rate-independent quasi-static regime (QS). With increasing excitation, granules can achieve a readily flowing phase. Like the molecules in a dilute gas, particles within this grain-inertial regime (GI) interact predominantly through short-duration, binary collisions (see Savage, Nedderman et al. 1983). Similar to an ideal gas within this dynamic state, GI stresses are quadratically rate-dependent. A
modified kinetic theory has been used to describe the underlying physics behind highly agitated granular materials (see Savage 1998). Between the two extreme regimes–GI and QS–however, there exists a transition flow that has yet to be empirically defined.

Studies have formulated both computational and theoretical results on the dynamics of granular flow (see Bagnold 1954; Savage 1984; Hanes & Inman 1985; Jop, Forterre, & Pouliquen 2006). Other authors have studied granular rheology and its dependencies on shear rate, volume-fraction, gravity, and applied pressure (see Campbell 1990; Karion & Hunt 1999; Hsiau & Shieh 2000; Tardos, Khan, & Schaeffer 1998; Tardos, McNamara, & Talu 2003; Bossis, Grasselli, & Volkova et al. 2004). Specifically, constitutive equations for shear and normal stresses have been hypothesized for granular hydrodynamics (see Savage 1998; Bocquet, Losert et al. 2002; Hendy 2005). However, the physics behind the transition between GI and QS regimes have mostly been overlooked, despite theoretical uncertainties (see Tuzun, Houlsby et al. 1982; Bocquet, Losert et al. 2002) and anomalous empirical data signaling its peculiarity (see Tardos, Khan, & Schaeffer 1998; Dalton, Farrelly et al. 2005). Particularly, recent experiments have indicated that both contact force distribution (see Corwin, Jaeger et al. 2005; Dalton, Farrelly et al. 2005) and geometrical anisotropy (see Majmudar & Behringer 2005) evolve within a sheared granular system under intermediate deformation rates. The collective effect of these local flow phenomena on the grain scale escalates the complexities of granular dynamics.

In this study we explore shear rates that span all three dynamic flow regimes: grain-inertial, transitional, and quasi-static. We perform laboratory experiments with both natural beach sand and milled quartz grains in a torsional shear rheometer where the rotational velocity is varied systematically over 5 orders-of-magnitude. We then compare the results from constant stress and constant volume configurations to address granular compressibility. For the constant stress experiments, volume is measured as function of strain-rate; for the constant volume experiments, normal and shear stress are measured as a function of strain-rate. To study the effects of flow anisotropy and force network, two different grain column heights are used. Sample sphericity is also observed to tremendously affect granular flow and result in an unexpected shear-weakening behavior. Moreover, a model is devised based on granular compressibility that retains Coulomb yield conditions and granular dilatancy. From fitting the model to the experimental data, we find consistent parameters in support of our equation-of-state for non-thermal, non-attractive particle systems. By doing so, we may have captured the experimental ramifications of localized flow phenomenon—the formation and collapse of clusters and force chains—as a self-organized resistance to granular deformation.

2. Dimensionless rate

Bagnold (1954) was the first to classify granular flows in a liquid medium. Savage and others later narrowed their efforts to dry systems and quantified their experiments with the Savage number (see Savage 1984),

\[ Sa = \frac{\rho D^2 \dot{\gamma}^2}{\sigma}. \]  

(2.1)

The Savage number is the ratio between inter-granular collisional stress and consolidation stress, and it is intended to delineate QS, GI, and transition flow regimes. Parameter \( \rho \) is particle density, \( D \) is mean grain diameter, \( \dot{\gamma} \) is shear rate, and \( \sigma \) is consolidating stress. Note the consolidation force includes all compacting forces such as gravity or electro-static forces if present.

From visually observing the experiment, we suspected that the shear band thickness—
Figure 1. Schematic of static and dynamic regimes of granular flow. (a) The static volume of column height $H_1$. Shaded and unshaded regions indicate volumetric void and solid fractions, respectively. (b) Shaded region shows the 'free' volume expanded from static packing. The total height is $H_2$ and the dilation constant is $\delta$. The exponential velocity profile has been observed previously. The effective void fraction $\phi$ is the ratio between the dynamic free expanding volume and the original static volume. Diagram is not drawn to scale.

the mobilized region of grains-remained independent of rate. Others have made similar observations for confined torsional flows (see Savage & Sayed 1984, Tardos et al. 2003, Karion & Hunt 1999). Visual measurements on the velocity profile are made at 50 rad s$^{-1}$ and 0.01 rad s$^{-1}$ recorded at 1000 fps and 1 fps respectively. Particle tracking data were fit to an exponential profile $u(y)$ of

$$u(y) = U_R \exp\left(-\frac{y}{L}\right),$$

where $U_R$ is the rim velocity and $L$ is the characteristic flow depth. The resulting fit at both velocities yields $L \approx 2D$, where $D$ is the mean grain diameter. In contrast, in free surface or avalanche flows driven by gravity the characteristic flow depth geometrical dependencies on channel width or tilt (see Jop, Forterre, & Pouliquen 2005).

From the rate-invariant characteristic depth of $\approx 2D$, we infer that the rate-independent shear flow thickness is of order $\sim D$. With this constraint, the Savage number (2.1) then becomes grain-size independent of the form

$$Sa = \frac{\rho U^2}{\sigma}. \quad (2.3)$$

3. Experiment setup and procedure

In the present experiment, we investigate the connection between stress, volume-fraction, and strain rate for a granular bulk over a wide range of deformation rates ($\sim5$ decades). Our objective is to understand the granular flow regimes, with an emphasis on the transition regime and its elusive rheological properties. Using a torsional rheometer shown in figure 2, we intend to study particle flows governed by purely repulsive and frictional interactions. The varying parameters in our experiment are top-plate height, angular velocity, and pressure, while material properties, i.e. rigidity and average grain size, are fixed. Dependencies on polydispersity and sphericity, although not substantially examined, are discussed in context of our model and interpretation.

To accurately analyze a system of discrete, mobilized particles, it is vital to account for the following inherent features of granular materials: compressibility, segregation, crystallization, and packing configurations. First, the compressible nature of the particle
'fluid' warrants two different experimental procedures: constant-volume and constant-pressure (see §3.2 for discussion). In contrast, compressibility is insignificant for the rheology of conventional liquids.

Second, segregation occurs within granular systems of different grain sizes (see Makse, Havlin et al. 1997; Shimbrot 2004). Polydispersity—grain mass variance—in our rotational device, in response, creates an inertial stress gradient in the radial direction. The stress gradient then segregates large grains to the rim away from smaller grains near the center. To investigate the impact of this segregation effect on our results, we performed the same experiments on beach sand with and without sifting.

Third, crystallization occurs within systems of spherical particles where abrupt changes occur at the liquid-crystal cross-over (see Drake 1990). To avoid this transition and to sustain an amorphous state at low or zero shear rates, therefore, the chosen polydisperse sample must have irregular, jagged shapes. Thus, we choose samples of angular beach sand and spherical F-35 foundry sand (US Silica Co.) to show sphericity effects on granular rheology. Figures 3 (a,b) and table 1 summarize the properties of the three samples–sifted sand, unsifted sand, and US Silica F-35–used in our experiments.

Last, there exist many meta-stable configurations, or stackings, of any given set of particles depending on the loading history (see Campbell 2005). These meta-stable packing configurations arise from the frictional nature of interacting grains, where the smallest perturbation in the magnitude or the direction of the compressive stress can disrupt this fragile arrangement (see Liu & Nagel 1998). One implication of this fragileness is material compaction: granular packing increases its solid volume-fraction when subjected to vibration or deviatoric strain (see Duran 1999). Thus in our experiment, we anneal each granular sample from a pre-experimental shear until compaction does not occur within the experimental time scale of hours.
3.1. Instrument setup and sample preparation

To investigate the granular flow, we use a torsional rheometer (AR-2000, TA Instruments Inc.). As seen in figure 2, the system has an upper rotating plate with 20 mm diameter and a lower fixture that can detect normal force (±0.1N) through an internal force transducer. The rheometer is a highly sensitive, feedback-controlled instrument that simultaneously monitors torque, normal force, top-plate height, and angular velocity.

Traditionally, the system is used to study the rheological behavior of conventional fluids. In our application, the upper plate is used for compression either by controlling normal stress (±1 Pa) or plate height (±1 µm), while shearing through a user-specified velocity range. The upper plate has taped 80-grit (300 µm sand grains) sandpaper to facilitate stress transmission. Tape is also set on the bottom fixture to gain traction.

To contain the grain sample, a Teflon self-lubricating sleeve, with a thickness of 0.4 cm is concentrically aligned around the shearing plate. Although the sidewall friction is unavoidable, the system can be rotationally mapped to counter all of the external resistance by applying a background torque. During optimal alignment, the experiment achieves an average sidewall stress contribution of ≈1% the total shear stress. An annulus setup with concentric inner and outer walls has been considered but all attempts failed to reduce wall friction to below 10% of the overall stress. Sidewall flex/stretching is not of concern for stress variations of ∼kPa for a concentric cylinder.

The samples used in these experiments are natural and sifted beach sand and US Silica F-35 foundry sand, all of which are composed of grains that are highly rigid and irregular in shape as shown in table 1 and figure 3 (a,b). The sifted beach sand is prepared by sieving the sample through a US Std. mesh size 18 and 120, respectively with 1000 µm and 125 µm openings. US Silica F-35 is sieved from factory. The beach sand is predominantly quartz (>70%) with small quantities of lithics and other minerals while the US Silica sample is entirely quartz. These materials have a Young’s modulus of 30 – 70 GPa and thus they are highly rigid compared to our experimental stresses of ∼10^3 Pa. Other relevant sample properties are listed in table 1. The reference volume-fraction is measured by weighing the samples after compaction with light tapping. Grains examined post-experimentally do not show changes in polydispersity and sphericity.

3.2. Procedure

To begin, the granular sample is loaded up to one of two heights: 1 mm and 6 mm that correspond to roughly 3 and 20 grain-diameters, respectively. Although there are large 1 mm grains in the 1 mm columns, shape irregularities and surface roughness ∼300 µm
allow smaller grains to contact the top and bottom surfaces. The reason for using the 3-grain column is, according to numerical simulations of Tardos, McNamara et al. (2003) and experimental observations by Hanes & Inman (1985), that 10 to 13 grain-diameter is the ‘effective’ shearing zone in a granular layer because of the exponential decay of the velocity with depth discussed in §2. The result from a 3-diameter column would therefore pin-point the effect of an inadequate shear zone; an effect of shallow inter-grain force-propagation on the stress-rate relationship (see de Gennes 1999; Majmudar & Behringer 2005).

After loading the granular sample, the granular layer is pre-sheared for ~100 rotations to make the packing consistent. Then during every experiment, the sample is sheared from fast to slow at logarithmically distributed velocities. Slow-to-fast experiments have similar results as will be shown in figure 6—the consistancy shows that the shearing surface does not degrade significantly during high shear-rates but after many runs, degradation eventually produce erratic results. During shear, the sampling rate for shear/normal stress and column height is 10 Hz. Other experimental parameters, i.e. velocity range and averaging times, are given in tables 2. Generally, the chosen velocity range stays above rates that result in stick-slip (see Aharonov & Sparks 2004) and below rates that promote excessive wear on the top-plate sand paper as shown in figure 3 (d).

To account for granular compressibility, two general shearing conditions are used: constant-volume and constant-pressure. In the constant-volume procedure, the specified column height stays fixed while the instrument varies pressure and rate. The specified column height is system-determined from a pre-experimental procedure of fixed pressure (10⁴ Pa) and velocity (10 rad s⁻¹). The reason is because of volume-dilatancy, the column height in the constant-volume procedure must be set while the grains are mobilized. In this configuration, granular volume-fraction remains constant. The second shearing condition is a constant-pressure configuration. A user-specified normal stress

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**Figure 3.** Micrographs of actual grains using KP-D50 (Hitachi) under 20X magnification. Samples used are (a) US Silica F-35 foundry sand, and (b) beach sand. The two samples shows large sphericity differences but with similar averaged size. Photographs using Canon PC1060 personal camera show (c) surface size segregation of unsifted sand after reaching steady-state and (d) shear surface wear near the edge of the 20 mm diameter sand-paper after ≈10 experiments.
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System parameters  Constant-volume  Constant-pressure
Column heights  1 to 12 mm  –
Pressures  –  1 to 7 kPa
Velocity step 1  0.001–0.01 rad s$^{-1}$ (300s)
Velocity step 2  0.01–1 rad s$^{-1}$ (180s)
Velocity step 3  1–100 rad s$^{-1}$ (30s)

Table 2. Control parameters of constant volume and constant pressure procedures. The column height is user-specified at 10 rad s$^{-1}$ for a particular pressure during constant volume procedures. The pressure is user-specified during constant pressure procedures. The velocity steps and their range are determined from optimized averaging time indicated in parenthesis. Each logarithmic decade has 6 to 10 data points.

($\sim$ kPa) is feedback-controlled by adjusting the column height, resulting in changes of volume-fraction as a function of shear rate. Both constant-volume and constant-pressure procedures have fixed numbers of particles.

4. Results and discussion

A torsional shearing device cannot accurately capture the rate-dependent rheology of non-Newtonian fluids. The drawback for having a large torsional strain is that the shearing velocity becomes a function of the radius, $\sim r$, measure from the center. In the context of granular flow, the issue becomes that the flow is always quasi-static in the center even if the rotational velocity is extremely high. The measured stresses is therefore an averaged stress $\langle \sigma \rangle$ given as

$$\langle \sigma \rangle = \frac{2\pi}{A_{disk}} \int_0^R \sigma(r)rdr.$$

We therefore need to approximate the Savage number in (2.3) with caution. We remedy this issue by computing an effective linearized velocity $U = \omega R^*$ using the stress-centroid to find $R^*$ as

$$R^* = \frac{\int_{disk} r (\tau \ dA)}{\int_{disk} \tau \ dA}.$$

(4.1)

where $r$ is radius and $dA$ is the infinitesimal disk area. Then Savage number from (2.3) becomes

$$Sa = \frac{\rho(\omega R^*)^2}{\sigma},$$

(4.2)

where $(2/3)R \leq R^* \leq (4/5)R$, $\omega$ is angular velocity and $R$ is the shear plate radius. Within the rate-dependent GI regime, the constant factor is $4/5$ rather than $2/3$ because the inertial stresses vary with radius as $\sim r^2$. Therefore the velocity (4.2) under-estimates the equivalent linear velocity, and thereby reduces Savage number, in the GI regime. The analysis also disregards the effects of secondary flow, with its velocity gradient in the radial direction. Hanes & Inman (1985) have used similar assumptions to interpret their torsional shear cell results.

4.1. Constant-volume experiment

Figure 4 outlines the flow regimes to be presented. The GI regime begins around $\omega = 25$ rad s$^{-1}$ and the conversion using (2.3) and (4.2) results in $Sa \sim 1$ at this velocity. The QS regime and its non-trivial transition into the GI regime is also observed. Figure 4
Figure 4. Log-log plot of shear stress versus Savage number of $\approx 1$ mm column. Both experiments have same column heights and similar pressures. Savage number is calculated at the transition into both grain-inertial and quasi-static regimes to be $S_a \sim 1$ and $S_a \sim 10^{-7}$, respectively.

is reminiscent of the powder flow diagram envisioned by Tardos et al. (2003) depicting various granular flow regimes. Using our parameters, the dimensionless rate used in Tardos et al. (2003) and Klausner et al. (2000) also results in $U/(gD)^{1/2} \sim 1$ and the consistency amongst the two dimensionless rates validates the present flow regimes.

4.1.1. Grain-inertial (GI) and Quasi-static Flow (QS)

In figure 5 (a), the shear stress is plotted against angular velocities through a range of $10^{-3}$ to $10^2$ rad s$^{-1}$. They correspond to Savage number on the order of $10^{-11}$ to 1. The two limiting regimes—GI and QS—can be easily identified for all three samples via visual inspection where the stresses are rate-independent and quadratically rate-dependent, respectively. To fit our results, a power-law is used

$$\tau = \tau_0 + \tau_1 \omega^n \quad \text{or} \quad \tau = \tau_0 + (\tau_1 C) \dot{\gamma}^n$$

(4.3)

where $\omega$ is the rotation rate, $\dot{\gamma}$ is the shear rate at the wall and $\tau_0$, $\tau_1$ and $C$ are rate-independent fitting parameters.

Our power-law fit yields a nearly quadratic stress-rate relationship similar to many experimental and theoretical results, showing a quadratic stress dependence on shear rate (see Savage 1984; Savage & Sayed 1984; Hanes & Inman 1985; Karion & Hunt 1999; Klausner, Chen et al. 2000). From figure 5 (a,b) insets, the rate-dependent shear stress within the grain-inertial regime for polydispersed beach sand is observed. The rate-dependence is nearly quadratic, i.e. $n \approx 2$, for both shear and normal stresses, but the lack of GI data results in large standard deviation values of $\sim 0.1$ computed from boot-strapping. The present experiment agrees with previous work that yielded a range of powers, $n$, between 0.75 and 2 for the grain-inertial stresses (see Savage 1984; Klausner, Chen et al. 2000; Sawyer & Tichy 2001; Tardos, McNamara et al. 2003).

All three flow regimes are revealed for a medium with a single volume-fraction. In contrast to the conclusion drawn by Campbell (2002), it is possible to obtain the GI and the QS regimes from an equally dense medium. In our case the volume-fraction of packed sand is $\approx 0.61$ and for polydispersed, non-spherical particles. It is slightly less than the
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maximum random close packing for spheres of ≈ 0.64–angular, polydispersed mixtures should have a slightly higher density (see Visscher & Bolsterl 1972; Yu, Zou et al. 1996).

4.1.2. Transitional Flow

In figure 5 (a,b), both shear and normal stresses are inversely proportional to shear rate within the transitional regime using unsifted and sifted beach sand. Three runs with sifted sand at average pressure ~14 kPa uses identical parameters to show consistency. One additional run with sifted sand at ~10 kPa shows the dependence of pressure: pressure induces shear-weakening since stresses dip less under smaller pressures. Unsifted sand at ~10 kPa shows minimal grain segregation effects where shear-weakening during intermediate rates matches well with sifted sand at the same pressure. At a higher sphericity, however, US Silica F-35 sheared at ~14 kPa do not weaken during the same transitional velocity range. Average pressures are based on the entire velocity range. In addition to figure 5, figure 6 shows repeatability of the current results for unsifted sand and F-35 as well as results from slow-to-fast shearing.

Here we offer a simple model to show that the shear-weakening phenomenon is not an artifact of the torsional geometry–namely the non-uniform shear rate that depends on radius–and that similar stress-rate relationship would occur in a linearized shear-flow. Following §4.1.1 and (4.3), we assumed that

$$\tau = \tau_0 + \tau_1 C \left( \frac{\omega R^*}{L} \right)^2, \text{ for } \dot{\gamma} \sim \frac{\omega R^*}{L},$$

where $R^*$ is derived in (4.1) and $L$ is the characteristic length from (2.2). By simply separating the total stress into frictional and kinetic parts, this model similar to previous work (e.g., Savage 1984) can only predict a monotonic increase $\tau$ with rate $\omega$. This is not what we observed. The shear-weakening transitional flow cannot be generated from (4.4), thus it must be intrinsic to the rheology of a granular fluid, rather than to any geometrical artifacts.

The exact physics behind the weakening is unknown. However, we offer a qualitative explanation: the observed pressure-induced shear-weakening in granular systems elucidates the evolution of its force network. As shown by Ostojic, Somfai, & Nienhuis (2006) that under large loads, the fractal medium adapts to external stresses by aligning strong contacts into filamentary force chains. This stress-induced anisotropy also exists geometrically (see Majmudar & Behringer 2005) where the filamentary chains physically align to the direction of the externally applied shear stress. These adaptive chains can then resist shear by carrying ≈10 times the mean external stress (see Corwin, Jaeger et al. 2005). Together, in response to stress, the above observations can be interpreted as a self-organizational stiffening of granular materials.

As rate increases, grain interactions become more collisional and less frictional (see Corwin, Jaeger et al. 2005). Instead of prolonged contacts, these rapid collisions reduce spatial anisotropy and hence decreases its resistance to shear. This force network breakdown is then illustrated by the shear-weakening transition. As seen in figure 5 (a,b), under constant volume, the dissimilar stresses within transitional regime seem to converge when approaching GI regime, for pressures of 10 and 14 kPa. This trend indicates the diminishing effect of contact force anisotropy with increasing shear; a consistent interpretation that leads to a more uniform (ergodic) kinetic GI regime where grains take on binary collisions. A similar shear-thinning rheology is common in dense, hard-core colloidal fluids (see Larson 1999).

Another crucial feature arises in the comparison between beach sand and US Silica F-35 where shear weakening is not observed for the latter (also in figure 6). In this
Figure 5. Log-log plot of wall stresses versus rate at constant volume and \( \approx 6 \) mm columns. (a) Shear stress \( \tau \) versus angular velocity \( \omega \). Average normal stresses \( \sigma_{\text{avg}} \) uses all pressure data of a single run. (b) Normal stress \( \sigma \) versus angular velocity \( \omega \). The insets (F-35 not shown) show the power-law exponent fitted to the data along with the coefficient \( n \) in (4.3). The standard deviation values are based on on 100 bootstrap trials. Sifted sand is used unless noted in the legend. Unlike beach sand, F-35 foundry sand of higher sphericity does not weaken in the transitional regime. The column height is between 6 mm and 12 mm for all runs which exceeds the mobilized regions of \( \approx 10 \) grain diameters; thus instead of column-height, averaged pressures label each run in the legend.

In general due to the 'caging' effect–mobility hindrance by neighboring particles–grains with lower sphericity flow less easily than spherical grain samples (see Yu, Zou et al. 1996; Zou and Yu 1996; Larson 1999). Highly spherical grains...
hence have a more ordered granular-fluid phase than highly angular grains. As seen in figure 5, under the same average pressures, lower shear and normal stresses are observed for F-35 than for beach sand. Flowing with perhaps local crystallized regions (see Drake 1990), the self-organized stiffening mechanism possible for angular beach sand is entirely avoided by spherical F-35. In hindsight, O’Hern et al. (2003) have also suggested that a mixture of highly angular particle is required to explore the entire granular phase space.

The above interpretations may explain the absence of the transitional regime from earlier simulations and experiments. Other concentric shear cells utilize gravity as the consolidating pressure on the order of $10^2$ Pa at mid-height for a 10 cm column (see Bocquet, Losert et al. 2002; Tardos, McNamara et al. 2003; MiDi 2004). Insufficient gravity forces, together with perhaps uncontrolled volume, may not induce the same structural responses explained here. Similarly, simulations using perfect spheres with varying rigidity can (see Campbell (2005); da Cruz et al. (2005); Lois et al. (2005)) only recover the rate-independent transition regime as indicated by our quasi-spherical F-35 samples.

The shear-weakening transition is shown to be almost insensitive to sample polydispersity by the sifted sand results in figures 5 and 6. This is consistent with a configuration where the stresses are transmitted vertically between relatively monodispersed particles. Driven by the rotating plate, segregation occurs by grains exerting centripetal forces directed outward in the radial direction. As shown in figure 3(c) during steady state, this outward inertial force circumferentially aligns equal-sized grains near the shearing surface with larger grains separate from smaller ones. As grains segregate by size, the colliding particles in adjacent horizontal layers are relatively mono-dispersed. Therefore a torsional cell may reduce, or even eliminate, the effect of polydispersity in steady-state granular rheology. The result supports the assumption of a monodispersed sample that is made to calculate Savage number in §2.

We note in passing that the lack of a shear-weakening dip in the US Silica F-35 results demonstrates that the dip is not an artifact of our torsional device.

Figure 6. Log-log plot shear stress versus angular velocity at constant volume. Similar to fast-to-slow experiments in figure 5(a), slow-to-fast experiments show transitional shear-weakening rheology. Samples F-35 and unsifted sand experiments shows repeatability in extension to figure 5(a).
Figure 7. Semi-log plot showing coefficient of friction versus Savage number for beach sand in constant-volume experiment. The plot compares experimental data from high and low shear column-heights. The higher height runs exceed the shear band thickness of ≈10 grain diameters. The lower height has ≈3 grain diameters across. \( \theta_{\text{repose}} \) is the angle-of-repose measured from the slope of a static granular pile. The observed increase of the friction coefficient \( \mu \sim \tan \theta_{\text{repose}} \) of all runs compared to the angle of repose of a static pile may indicate the dependence of friction coefficient on volume-fraction. All other runs exhibit regime-invariant friction coefficient.

4.1.3. Friction

In figure 7, we use the constant-volume method to investigate the friction coefficient by plotting the ratio from shear and normal stresses against Savage number. Not surprisingly, the close resemblance of shear and normal stress behavior produce a friction that is nearly independent of shear rate. In support of our result, both Savage (1998) and Cheng & Richmond (1978) proposed normal stress is proportional to the shear stress and independent of how particles interact.

In figure 7, the friction coefficient is compared with samples' angle of repose, \( \theta_{\text{repose}} \), measured from the inclination of a prepared sand pile. The resulting friction coefficient of \( \mu = 0.78 \), where \( \mu = \tan \theta_{\text{repose}} \), is about 11% lower than the compressed sand from our shear flow experiment (see Nedderman 1992). The comparison highlights the mobility hindrance due to confinement where grains under constant volume has less interstitial spaces than flowing grains near a free surface. The lack of space inhibits grain mobility, thus producing a higher friction coefficient than unconfined flows—the difference is revealed by the intricate filamentary network of force-chains explained in §4.1.2.

4.2. Constant-pressure experiment

Figure 8 is a semi-log plot of normalized volume-fraction versus rotation rate from constant-pressure experiments. It is erroneous to report the bulk volume-fraction since only the height change is measured. Thus the normalized volume-fraction is calculated by comparing height change to the characteristic shear band thickness, \( L \approx 2D \), discussed in §2 (see appendix B for derivation). From the observations, the volume-fraction data can also be separated into QS, GI, and transition regimes. As the mirror-image to the constant volume stress, it suggests that inverse Bagnold scaling, i.e. the quadratic rate dependence, also applies for volume-fraction. Moreover, the different peaks in figure 8
Figure 8. Semi-log plot of dimensionless volume-fraction, $\frac{\nu}{\nu_0}$, versus rotation rate; $\nu/\nu_0$ is calculated from the height measurements in Appendix B. Grain columns are approximately 6mm. The right-hand y-axis is the observed change in height between the plates. The maximum height-changes measured from low to high pressures are 15, 31, and 47 $\mu$m, respectively. The rising tails during high velocities occur due to low applied pressures from system. Note that the curves on this figure have the same general form as in figures 5 and 10 except that the y-axis is inverted (See § 5 for discussion in terms of constant compressibility).

shows that volume-fraction is inversely proportional to pressure, implying the existence of granular compressibility.

Constant-pressure and constant-volume rheologies are intimately related. The extremums in applied stresses and volume-fraction in figures 5 $(a,b)$ and 8 occur at the same rotation rate $(\omega \approx 23 \text{ rad s}^{-1})$. Under a non-equilibrated steady-state, the work done at the shearing surface dissipates completely through inelastic particle collisions. The work of surface stresses, given by $\nabla \cdot (\hat{T} \cdot \hat{U})$ where $\hat{T}$ is the total stress tensor, is directly proportional to the applied stress at the wall. Hence in figure 5 $(a,b)$, the minimum in shear and normal stresses is also the minimum in energy dissipation. Since volume inversely relates to pressure, a local minimum in stress corresponds to a local maximum in volume-fraction and vice versa. Thus in figure 8 with pressure kept constant, the maximum in volume-fraction occurs at the shearing state which corresponds to the minimum dissipation. The coincidence of extremums suggests that volume-fraction, stress, and strain rate are intimately linked through an equation of state that does not allow them to vary independently. Furthermore, the stress minimum and volume-fraction maximum indicate an optimum condition for transporting and manipulating granular systems.

4.3. Column height dependence

Figure 9 plots shear stress versus rotation-rate to compare results from 7 mm and 1 mm columns. The comparison shows the effect of inadequate sample size since the shear band depth captured at 0.01 and 50 rad $\text{s}^{-1}$ is 10 grain-diameters, or $\approx 4$ mm (see § 2). Similar to the results of 7 mm columns, shear stress as a function of rate for 1 mm columns also indicate clear distinctions of QS, GI, and transitional regimes.

The qualitative comparison indicates a close resemblance between QS and GI stresses from using two different column heights. The lack of grains, however, must affect the
Figure 9. Log-log plot of shear stress versus rotation rate. Plots compare results from 7 mm and 1 mm columns to show the effect of inadequate sample volume. The shear band depth captured at 0.01 and 50 rad s\(^{-1}\) is 10 grain-diameters, or \(\approx 4\) mm. Results from 1 mm columns indicate clear distinction of QS, GI, and transitional regimes. The relationship between rate and stress, since the flow-depth in the 1 mm columns is much less than the observed shear band thickness \(\approx 4\) mm. The granular fluid must cope with its thinner column with a higher anisotropy in force chain orientations as discussed in § 4.1.2. Therefore we conclude, in agreement with dimensionless analysis (see Appendix A), that the limiting regimes follow the basic frictional and inertial grain interactions regardless of flow thickness. Yet, 1 mm flows have dissimilar shear-weakening transitions than 7 mm flows, a claim supported in § 5.1 and Table 3 by the disparity between fitting coefficients.

5. Constitutive law

A key uncertainty in describing the fluid-like behavior of granular materials is the transition between the collisional and frictional regimes. The inability of the continuum constitutive relations, such as the Navier-Stokes equation, to account for the heterogeneities of granular flow has been a difficult issue to overcome. To model the macroscopic quantities of granular mediums, such as stress, packing density, or flow profile, many theories rely on the visco-plastic or elasto-plastic description used for metals (see Dartevelle 2004; Savage 1998). Although these approaches capture certain aspects of the frictional nature of granular materials by assuming perfect plasticity, they are inadequate for the results presented here. The rate-dependence in stress and volume-fraction captured in figures 5 and 8 indicate the complexities of granular unaccounted for by the continuum models.

To capture the macroscopic effect of localized flows, we describe granular flow from analogous concepts in conventional thermodynamics. Similar to pressure, temperature, and density for describing solids and fluids, dry granular flow may be defined by pressure, deviatoric strain rate, and volume-fraction (see Liu & Nagel 1998). Although granular fluids are non-thermal in the classical sense unlike molecular fluids, the minimization in gravitational potential nonetheless drives granular compaction under external excitations.
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into energetically favorable states. Below we develop an equation-of-state to model our results.

Our model must satisfy the shear-weakening and shear-compacting observations in figures 5 (a, b) and 8. Note, the following analysis only applies to beach sand results. Quantitatively, the two rate-dependent observations from our beach sand experiments within intermediate shear rates are

\[
\left( \frac{\partial \sigma}{\partial \nu} \right)_{\dot{\gamma}} > 0 \quad \text{and} \quad \left( \frac{\partial \sigma}{\partial \dot{\gamma}} \right)_{\nu} < 0, \tag{5.1}
\]

where \( \sigma \) is normal stress, \( \nu \) is solid-volume-fraction, and \( \dot{\gamma} \) is shear rate. The first inequality of (5.1) is the compressible relation that describe granular shear bands. It resembles a conventional compressibility relation. The physical explanation of the second inequality in (5.1) remains unknown, yet it agrees with the scenario presented in § 4.1.2 and figure 5.

Changes in volume-fraction with respect to shear rate can be derived from the two relations in (5.1) using the cyclic rule,

\[
\left( \frac{\partial \sigma}{\partial \nu} \right)_{\dot{\gamma}} \left( \frac{\partial \nu}{\partial \dot{\gamma}} \right)_{\sigma} \left( \frac{\partial \dot{\gamma}}{\partial \sigma} \right)_{\nu} = -1. \tag{5.2}
\]

Thus similar to the thermal expansion relation—if shear rate takes on the role of temperature—volume-fraction at constant pressure relates to rate by

\[
\left( \frac{\partial \nu}{\partial \dot{\gamma}} \right)_{\sigma} > 0, \tag{5.3}
\]

just as seen in figure 8.

To proceed, we define an observable compressibility \( \beta \) for the shear band within a two-phase, mobilized granular system.

\[
\beta \equiv \frac{1}{V} \left( \frac{\partial V}{\partial \sigma} \right)_{\dot{\gamma}}, \tag{5.4}
\]

where \( V \) is the volume of the test cell. Since \( V \propto \nu^{-1} \), changes in the cell volume can be mapped to changes in the solid volume fraction in the shear band, even though the total volume of the shear band cannot be measured directly. The compressibility is related to the data using the cyclic rule (5.2) as

\[
\frac{1}{V} \left( \frac{\partial V}{\partial \sigma} \right)_{\dot{\gamma}} = -\frac{1}{V} \left( \frac{\partial V}{\partial \dot{\gamma}} \right)_{\sigma} \left( \frac{\partial \dot{\gamma}}{\partial \sigma} \right)_{V}. \tag{5.5}
\]

Using (5.5) and the variations with strain-rate of the stress (\( \sigma \)) and column-height (\( \Delta H \)) data in figures 5 and 8, the mean granular fluid compressibility \( \beta \) in both quasi-static and transitional regimes is \( \approx 4 \times 10^{-3} \text{Pa}^{-1} \), independent of strain-rate.

We must clarify that the actual volume change of rigid constituent is negligible under low to moderate pressure, i.e. \( \sigma \ll E \), where \( E \) is the grain elastic modulus. Therefore, the compressible nature of granular flow is solely due to the volumetric changes in the interstitial spaces. In a crystalline solid, the physical interpretation of \( \beta \) lies in the second derivative of lattice potential per molecule with respect to lattice spacing. In granular flow, however, the physical origin of \( \beta \) is not yet understood.

Also, the compressibility is distinctly different than the irreversible process of granular consolidation (see Nedderman 1992; Nowak, Knight et al. 1997). For both static and dynamic granular systems, consolidation irreversibly reduces void fraction. Here, in contrast, we assume the changes in volume-fraction are entirely reversible. In this context,
the term reversible does not imply an isentropic process as in classical thermodynamics. Figure 8 shows volumetric reversibility as the same medium contracts and expands across a maximum volume-fraction.

To develop a constitutive law from a free-volume analysis, we now define an effective interstitial free-volume fraction $e$. Up to and excluding the GI deformation rates, the compressibility of a mobilized granular solid in terms $e$, at constant, non-zero shear rate is

$$
\frac{1}{e} \frac{\partial e}{\partial \sigma_{QS}} \dot{\gamma} = -\beta \quad \text{for} \quad \beta \geq 0,
$$

(5.6)

where $\sigma_{QS}$ is the contribution to the isotropic compressive stress in the QS and transitional regimes. (GI contribution is added later in (5.21)). The interstitial void fraction, $e$, is defined as the ratio between the expanded ‘free’ volume, which allows for grain mobilization, to the bulk static volume (see figure 1). It equals zero for a static packing but it has a finite value for a network of moving grains; $e$ relates to the granular volume-fraction $\nu$ as

$$
\nu = \frac{\nu_{\infty}}{1 + e} \quad \text{and} \quad e = \frac{\nu_{\infty} - \nu}{\nu}, \quad e \geq 0,
$$

(5.7)

and $\nu_{\infty}$ is the maximum volume-fraction during steady-state shear as a function of maximum pressure and minimum shear-rate. The effective void fraction is also a measure of grain mobility within a dense packing. Integrating (5.6) at constant compressibility and substituting in (5.7) at a constant shear rate $\dot{\gamma}$, yields

$$
\sigma_{QS}(\nu) = \frac{1}{\beta} \left[ \ln \left( \frac{\nu}{\nu_{\infty} - \nu} \right) + \text{Const} \right].
$$

(5.8)

Separating the integration constant into rate-independent and rate-dependent parts $A$ and $B(\dot{\gamma})$ respectively, (5.8) becomes

$$
\sigma_{QS}(\nu, \dot{\gamma}) = \frac{1}{\beta} \left[ \ln \left( \frac{\nu}{\nu_{\infty} - \nu} \right) + A + B(\dot{\gamma}) \right].
$$

(5.9)

To solve for the unknowns in (5.9), we use the inequalities in (5.1) to formulate the effect of volume-fraction on pressure and shear rate. From observation, volume-fraction $\nu$ has the following boundary conditions for the QS component:

$$
\nu = \nu_0', \quad \text{as} \quad \sigma \to 0, \dot{\gamma} \to 0 \quad \text{(5.10a)}
$$
$$
\nu = \nu_{\infty}', \quad \text{as} \quad \sigma \to 0, \dot{\gamma} \to \infty \quad \text{(5.10b)}
$$
$$
\nu = \nu_{\infty}, \quad \text{as} \quad \sigma \to 0, \dot{\gamma} \to \infty \quad \text{(5.10c)}
$$

where $\nu_0'$ and $\nu_{\infty}'$ are the maximum and minimum volume-fractions at the limit of zero applied load, respectively, and $\nu_{\infty}$ is the absolute maximum dynamic volume-fraction. However, in actuality for the current experiment, $\nu_0'$ and $\nu_{\infty}'$ are the respective absolute maximum and minimum volume-fractions for systems under gravity. For comparison, $\nu_0 ' < \nu_{\infty}' \leq \nu_{\infty}$ and all of them are quantities measured in a mobilized particle system. Note, although there is no absolute minimum packing under zero applied stress, gravity enforces the lower limit on volume-fraction in our experiment. Using boundary condition (5.10b), (5.9) becomes for $\dot{\gamma} \to \infty$

$$
\frac{1}{\beta} \left[ \ln \left( \frac{\nu_{\infty}'}{\nu_{\infty} - \nu_{\infty}'} \right) + A + B(\dot{\gamma}) \right] = 0
$$

(5.11)
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and if $B(\dot{\gamma} \to \infty) = 0$, then

$$A = \ln \left( \frac{\nu_{\infty} - \nu'_{\infty}}{\nu'_{\infty}} \right).$$

From here, (5.9) becomes

$$\sigma_{QS}(\nu, \dot{\gamma}) = 1 - \frac{1}{\beta} \left[ \ln \left( \frac{\nu_{\infty} - \nu'}{\nu_{\infty} - \nu} \right) + B(\dot{\gamma}) \right] \approx 1 - \frac{1}{\beta} \left[ \ln \left( \frac{\nu_{\infty} - \nu'}{\nu_{\infty} - \nu} \right) + B(\dot{\gamma}) \right]$$

(5.12)

where the approximation is applicable if volume-fraction changes are small so that $\nu/\nu'_{\infty} \approx 1$. We assume that volume-fraction is exponentially dependent on strain-rate with fitting constants $C_1$ and $C_2$, while satisfying the boundary conditions. According to shear-compaction in (5.3), the volume-fraction increases with shear rate of the form

$$\nu(\sigma, \dot{\gamma}) = \nu_{\infty} - \frac{\nu_{\infty} - \nu'}{1 - C_1 \exp (-C_2 \dot{\gamma})} \func(\sigma_{QS}),$$

(5.13)

where $\func(\sigma_{QS})$ is an arbitrary stress function to make equation self-consistent with (5.12). Note, the rigorous derivation of the exponential function of shear rate is not included in the current derivation. The physical meaning of $C_2 = f \left[ \rho U^2 / \sigma, \mu_f, \nu, e, s \right]$ may be the consolidation time $\tau_c$ given in Appendix A.4. The ratio $\tau_c / \dot{\gamma}^{-1}$ would then compare grain-settling time to grain-contact time. A larger $\tau_c$ describes a denser medium at a given pressure and particle-density, where grains move sluggishly into neighboring vacancies.

Substituting for $\nu$ using (5.13), (5.12) becomes

$$\frac{\nu_{\infty} - \nu'}{\nu_{\infty} - \nu} = \frac{1 - C_1 \exp (-C_2 \dot{\gamma})}{\func(\sigma_{QS})} = \exp [\beta \sigma_{QS} - B(\dot{\gamma})]$$

(5.14)

and solving for $\func(\sigma_{QS})$ and $B(\dot{\gamma})$, we get

$$\func(\sigma_{QS}) = \exp (-\beta \sigma_{QS})$$

(5.15)

and

$$\exp [B(\dot{\gamma})] = \frac{1}{1 - C_1 \exp (-C_2 \dot{\gamma})}.$$

(5.16)

Then using (5.15), (5.13) becomes

$$\nu(\dot{\gamma}, \sigma_{QS}) = \nu_{\infty} - \frac{\nu_{\infty} - \nu'}{1 - C_1 \exp (-C_2 \dot{\gamma})} \exp (-\beta \sigma_{QS}).$$

(5.17)

For completeness, by applying boundary condition (5.10a), the gravity-induced minimum volume-fraction is

$$\nu(\dot{\gamma} \to 0, \sigma \to 0) = \nu'_0 = \nu_{\infty} - \frac{\nu_{\infty} - \nu'}{1 - C_1}.$$  

(5.18)

and

$$\nu'_{\infty} / \nu_{\infty} \leq C_1 < 1$$

(5.19)

since $\nu \geq 0$ and $\nu \leq \nu'_{\infty} < \nu_{\infty}$. The magnitude of $C_1$ is a function the compacting forces, i.e. external pressure or gravity. As rate approaches zero, the limiting volume fraction $\nu'_0$ under zero gravity is referred to by Onoda & Liniger (1990) as the “dilatancy onset”—a particular packing-density where shear-dilatancy does not occur. From (5.17), the bulk stresses of a granular solid is
Figure 10. Log-log plot comparing data and model for pressure versus shear rate from the constant-volume experiments. The fit has two sets for 1 mm and 6 mm experiments. The fitting set includes: (1) Point-dash line uses the QS description (5.20). (2) Dash line uses the GI description (5.21). (3) Solid line is the sum of the previous two contributions (see 5.22). Other results totaling 5 runs each from 6 mm and 1 mm experiments show consistent fits.

\[ \sigma_{QS}(\nu, \dot{\gamma}) = \frac{1}{\beta} \ln \left[ \frac{\nu_\infty - \nu'_\infty}{\nu_\infty - \nu - C_1 \exp(-C_2 \dot{\gamma})} \right]. \]  
\[ (5.20) \]

Notice (5.20) only accounted for the states within the QS and the transitional regimes. To include the GI regime fit, we looked at the dimensionless groups of (A 5) where the stresses are quadratically dependent on shear rate. Therefore, similar to the Savage (1998) analysis, the GI stress is

\[ \sigma_{GI} = func(\nu, e, s) \rho D^2 \dot{\gamma}^2 = C_3 \rho D^2 \dot{\gamma}^2. \]  
\[ (5.21) \]

Here, the function term is reduced to a fitting constant \( C_3 \) since the sample and volume-fraction are kept constant in the experiment. Summing (5.20) and (5.21), we form a constitutive law for the observations in figures 5 (a,b) and 8, namely,

\[ \sigma_{sum}(\nu, \dot{\gamma}) = \frac{1}{\beta} \ln \left[ \frac{\nu_\infty - \nu'_\infty}{\nu_\infty - \nu - C_1 \exp(-C_2 \dot{\gamma})} \right] + C_3 \rho D^2 \dot{\gamma}^2. \]  
\[ (5.22) \]

As for shear stress, by assuming a constant proportionality that \( \mu = \tau / \sigma \), it becomes clear that

\[ \tau_{sum}(\nu, \dot{\gamma}) = \mu \left\{ \frac{1}{\beta} \ln \left[ \frac{\nu_\infty - \nu'_\infty}{\nu_\infty - \nu - C_1 \exp(-C_2 \dot{\gamma})} \right] + C_3 \rho D^2 \dot{\gamma}^2 \right\}. \]  
\[ (5.23) \]

Also, a composite volume-fraction \( \phi \) that relates to the ratio of the interstitial volumes \( e \) and \( e'_\infty \) is given as

\[ \phi = \frac{e'_\infty}{e} \approx \frac{\nu_\infty - \nu'_\infty}{\nu_\infty - \nu}. \]  
\[ (5.24) \]
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5.1. Least-square Fit

In figure 10, the log-log plot fits (5.22) to the constant-volume experiment in figure 5 (b). The pressure versus strain rate data is for sifted beach sand. The log-log plot shows the stress contribution from the GI factors in (5.21) and the QS factors in (5.20) regimes separately, and their sum is in good agreement with the observed flow regimes. In figure 11, the log-log plot fits (5.25) to the constant-pressure experiment in figure 8. The interstitial volume ratio \( \phi \), as given in 5.24, is found using the measured column height with the method described in Appendix C.

The constants \( \beta, C_1, C_2 \) and \( C_3 \) are found using an iterative least-square fitting procedure in MATLAB. Table 3 tabulates the fitting constants for experiments of 1 \( \mu \)m, and 6 \( \mu \)m, columns as well as constant-volume and constant-pressure configurations. The resulting fits correlate well only among the 6 mm experiments. The coefficients for 1 mm parameters are significantly different than the 6 mm fits. The disparity indicates the lack of scale-invariance with respect to system size for the fitting parameters \( \beta, C_1, C_2 \) and \( C_3 \). Thus based on the lack of consistency we speculate that our constants are only consistent for sufficiently large sample volumes (see § 4.3).

Table 3 summarizes our fitting results. The a priori value of 13 \( \mu \)m is used for the dilatancy height \( \delta \) in (C 5) to calculate the interstitial volume ratio \( \phi \) for all fits. Compressibility \( \beta \) can be compared to the mean value \( \approx 4 \times 10^{-3} \) Pa\(^{-1} \) measured from (5.5) using stress-rate and volume-rate data. The similar values in both fit and model confirm

---

**Figure 11.** Log-log plot of comparison between theory and free-volume ratio versus shear rate from the constant-pressure experiment. Grain column is 6mm. The fitting set includes: (1) Point-dash line uses the QS description. (2) Dash line uses the GI description. (3) Solid line is the product of the previous two contributions as shown in (5.25).

The interstitial free-volume ratio \( e \) and the minimum ratio \( e'_{\infty} \), which is applicable as the rate approaches infinity, are based on relations in (5.7). Appendix C calculates the experimental \( \phi \) based on measure height change. Thus, (5.22) can be written as

\[
\phi = [1 - C_1 \exp(-C_2 \dot{\gamma})] \exp \left[ \beta \left( \sigma_{\text{sum}} - C_3 \rho D^2 \dot{\gamma}^2 \right) \right].
\]  

(5.25)
Table 3. Summary of the fitting constants for constant-volume and constant-pressure experiments. All values are averaged amongst all experimental fits. The ratio compares the parameters fit between the 6 mm constant-volume and constant-pressure data. A ratio of unity signifies the consistency between the fitting parameters across both experiments. The ± values represent standard deviations of the parameters for the best-fit to each of the experimental runs under the given conditions. $C_1$ is approximately unity but it must satisfy the constraint of $C_1 < 1$ for materials under consolidation. Averaging involves 5 runs each for 1 mm and 6 mm constant-volume experiments and 2 runs for constant-pressure experiment.

<table>
<thead>
<tr>
<th>Fitting parameters</th>
<th>CG$<em>{1</em>{mm}}$</th>
<th>CG$<em>{6</em>{mm}}$</th>
<th>CF$<em>{7</em>{kPa}}$</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta \times 10^4$ (Pa$^{-1}$)</td>
<td>15 ± 4</td>
<td>6.1 ± 0.4</td>
<td>5.9 ± 0.3</td>
<td>1.0</td>
</tr>
<tr>
<td>$C_1$</td>
<td>$\approx 1$</td>
<td>$\approx 1$</td>
<td>$\approx 1$</td>
<td>1.0</td>
</tr>
<tr>
<td>$C_2 \times 10^5$ (s)</td>
<td>19 ± 6</td>
<td>2.1 ± 0.8</td>
<td>2.5 ± 0.6</td>
<td>0.84</td>
</tr>
<tr>
<td>$C_3 \times 10^3$</td>
<td>13 ± 6</td>
<td>0.9 ± 0.7</td>
<td>2.1 ± 0.5</td>
<td>0.42</td>
</tr>
<tr>
<td>$\delta$ (µm)</td>
<td>13</td>
<td>13</td>
<td>13</td>
<td>1.0</td>
</tr>
</tbody>
</table>

† The same value for the dilatancy volume, $\delta$, is imposed for all experiments a priori to solve for $\phi$ in Appendix C.

the compressible nature of granular fluids. Constant $C_1$ is nearly unity; it is restricted to be less than 1 in the fitting procedure as required for non-negative volume-fraction as derived in (5.19). $C_1$ determines the solid volume-fraction at the "dilatancy onset" (see Onoda & Liniger 1990) and it is a positive function of consolidation pressure. Constant $C_2$ is $\sim 10^{-5}$ s, and for the parameters in Table 1, the settling time $\tau_c$ is $\sim 10^{-4}$ s from (A 7). The near-coincidence of these values suggests a line of investigation for further research. Constant $C_3$ is a function of volume-fraction according to (A 5) and the analysis by Savage (1998). The slight inconsistency between constant volume and constant pressure values of $C_3$ in table 3 support this claim.

The agreement between our analytical model and empirical data from both heights, along with similar fitting parameters, validates the present model. Interestingly, the compressibility of air at standard temperature and pressure (STP) condition is $10^{-5}$ Pa$^{-1}$, about one order of magnitude less than the compressibility of flowing sand.

6. Conclusions

In our attempt to recover the transitional granular flow regime, we have seen a glimpse of how particles collectively respond to simple shear. The five decades of shear rate in the present experiment correlates to many natural and industrial processes: avalanches, landslides, dredging, and particle fluidization in mixing, segregation, and compaction. Our results show:

(i) During constant volume, shear-weakening occurs during transitional flow rates. Both shear and normal stresses 'dip' and reach a minimum value. Bagnold scaling $\sim \dot{\gamma}^2$ is observed for the grain-inertial regime.

(ii) During constant pressure, shear-compaction occurs during transitional flow rates. The reversible volume-fraction plateaus and then decreases with inverse Bagnold scaling $\sim \dot{\gamma}^{-2}$ in the grain-inertial regime.

(iii) The transitional flow regime is much broader than previous observations spanning a Savage number range from $10^{-7}$ to $10^{-1}$.

(iv) An equation-of-state devised based on granular compressibility unifies our find-
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ings. Fitting parameters remain consistent from all experiments with adequate shear-band thickness.

(v) Particle sphericity has a tremendous effect on transitional regime. Shear-weakening does not exist for spherical grains, only rate-independent Coulomb stresses that precedes Bagnold stresses $\sim \dot{\gamma}^2$ above Savage number $\sim 1$.

The present constitutive law provides insight into the scenario presented in § 4.1.2 illustrating a plausible physical origin for the observed granular rheology. As the strain-rate increases in a dense flow, force chains collapse and the granular fluid becomes less stiff. The weaker fluid requires smaller normal forces to maintain the packing imposed by the constant volume boundary conditions. The result is the 'dip' that is observed in our experiments.

The quantitative analysis has its shortcomings. In particular, it does not consider material properties such as the coefficient of restitution, rigidity, and contact friction. We do, however, suspect inelastic collisions, as well as boundary effect, to be a part of a separate conservation law that balances the inter-granular momentum and energy (see Jop, Forterre et al. 2005), a law of motion that resolves velocity profile, granular temperature, dissipation, and convection (see Knight, Ehrichs et al. 1996).

Despite being an incomplete description of granular flow, the current model delineates granular flow regimes. Moreover, the compressible model formulated for the transitional regime may reflect the emerging macroscopic effect of the self-organizational network of force chains. Although the local contact forces and volume-fraction are highly probabilistic (see Behringer, Howell et al. 1999), spatial and temporal ensemble averaging of the physical quantities provides a deterministic granulodynamics as indicated by our experiment.

We hope the presented model can predict granular rheology in other configurations. A nontrivial extension, however, is needed to account for disparate methods of excitation. The shear-driven deformation in our case is unlike gas or vibration-induced fluidization. It will be interesting to see a comparison with the results from a chute flow or fluidized bed. The comparison may further elucidate the exact physical meanings of the fitting constants $C_1, C_2, C_3$ used in our equations. The difficult interpretation of the heterogeneous shear band must be reconciled in the various flow types.

Our future work also includes a new annulus setup. A confined channel will help to eliminate any possible effect biased to the shear rate variation of our current setup. Material confinement and excessive wear are the difficulties to overcome for the current annulus geometry. Preliminary results show good agreement with the present flat-plate configuration.

Appendix A. Dimensionless analysis

A general functionale for the macroscopic granular flow shear stress in term of other independent variables is

$$\tau = fn(\sigma, \rho, U, g, \nu, D, D', e, \mu_f, E, s). \quad (A1)$$

where $\sigma$ is the applied pressure, $\rho$ is solid density, $U$ is translational grain velocity, $D$ is the average grain diameter, $D'$ is the sample size variance, or the degree of polydispersity, $g$ is the acceleration of gravity, $e$ is the coefficient of restitution defined as the ratio between incident and reflected velocities, $\nu$ is the bulk solid volume-fraction, $\mu_f$ is the inter-grain friction coefficient, $E$ is the bulk elastic modulus of the material, $s$ is the grain sphericity of which affects inter-particle friction. Equation A1 is similar to that offered by Savage (1984). Since shear band width is proportional to grain size, the fluidized shear
thickness is implicitly included in the analysis. Below we quantify each regime using the Buckingham Pi theorem.

A.1. Quasi-static regime

To understand the pertinent parameters within the quasi-static regime, we choose applied pressure $\sigma$, particle density $\rho$, and gravity $g$ to quantify the flow. Thus (A 1) in terms of dimensionless groups for the quasi-static regime is

$$\frac{\tau}{\sigma} = fn\left(\frac{\rho U^2}{\sigma}, \frac{\rho g D'}{\sigma}, \frac{E}{\sigma}, \mu_f, \nu, e, s\right).$$ (A 2)

In our analysis for large applied loads, the gravitational body forces can be neglected in (A 2). The particle Young’s modulus, $E$, is also assumed to be sufficiently high compared to the applied pressure, $\sigma$. Also due to segregation, $D' \ll D$. From the above assumptions, for vanishing shear velocity and thus neglecting inelastic collision effects, (A 2) becomes

$$\frac{\tau}{\sigma} = fn(\mu_f, \nu, s).$$ (A 3)

The resulting stress behavior is only a function of particle friction and sphericity, and volume-fraction. The overall macroscopic properties are also rate-independent for a granular layer operating in the quasi-static regime. The effect of grain sphericity $s$ may tie directly into volume-fraction as discussed in §3.

A.2. Grain-inertial regime

In a rapid shear flow, instead of a frictional sliding contact between grains, it is the inelastic collisions that are responsible for most of the momentum transport. The solid concentration is low and particles have a random fluctuation velocity component in addition to the mean velocity (see Savage 1984). For the GI regime, we choose $\rho$, $D$, and $U$ to quantify the flow. Thus (A 1) in its dimensionless groups for Savage numbers greater than unity becomes

$$\frac{\tau}{\rho U^2} = fn\left(\frac{\sigma}{\rho U^2}, \frac{D'}{D}, \frac{E}{\rho U^2}, \frac{g d}{U^2}, \mu_f, \nu, e, s\right).$$ (A 4)

Analogous to the procedure done in §A.1, the particle-kinetic quality of granular material is revealed by neglecting gravitational and lower order velocity terms in (A 4). Also due to segregation, the medium becomes mono-dispersed circumferentially so that $D' \ll D$. The resulting mean shear and normal stresses, $\tau$ and $\sigma$, are described as

$$\frac{\tau}{\rho U^2} = fn\left(\frac{\sigma}{\rho U^2}, \nu, e, s\right).$$ (A 5)

Stresses are proportional to the square of the average grain velocity, or the granular kinetic energy. (A 5) is comparable to the relations given in Bagnold’s analysis (1954).

A.3. Transitional flow regime

In the limiting flow regimes discussed above, bulk stresses are controlled by different mechanisms and physical parameters as mentioned in Appendices A.1 and A.2. Therefore, the transitional regime between them should consequently exhibit features that resemble both rate-dependent and rate-independent regimes. Depending on the relative stress levels and packing density and assuming the compressible nature of granular packing density, various dimensionless groups in (A 3) and (A 5) can be significant. We anticipate
the importance of first order velocity to the quasi-static groups. Hence,

\[ \frac{\tau}{\sigma} = f_n \left( \frac{\rho U^2}{\sigma}, \mu_f, \nu, e, s \right). \] (A.6)

### A.4. Consolidation time-scale

The settling or relaxation time for granular fluids under pressure can be approximated from kinematics as \( \tau_c \sim \sqrt{2l/\bar{a}} \). Length scale \( l \sim D \), the mean grain diameter, represents the vacancy length per grain that allows for grain mobility, as a direct result of shear-dilation–interstitial volume expansion due to granular deformation (see Reynolds 1885). It resembles the length scale Bagnold derived from solid volume-fraction (see Bagnold 1954). Average grain acceleration \( \bar{a} \) from wall collisions relates to the vacancy length, consolidation pressure \( \sigma \), and grain density \( \rho \) by \( \bar{a} \sim \sigma/\rho l \). Similarly for grains rotating from wall shear, angular acceleration becomes \( \bar{\alpha} \sim \tau/\rho D l \) where \( \bar{\alpha} \sim \bar{a}/D \). These rough estimates also assume Hertzian contact area as \( \sim D^2 \) from a rough shearing surface. Finally since \( l \sim D \) and \( \sigma \sim \tau \), we arrive at an average relaxation time,

\[ \tau_c \sim D \sqrt{\frac{\rho}{\sigma}}. \] (A.7)

The time scale (A.7) refers to how fast grains settle into their respective vacancies under compression. It implicitly assumes only for dense flows, so that \( l \sim D \) becomes small to a degree where the system no longer consolidates relative to the experimental time scale \( \sim \dot{\gamma}^{-1} \). It is at this dense packing where the system can approach steady-state shear. Also note for extremely low shear rates, the above analysis is only valid for angular grain mixtures where crystallization–the ordered packing of a lattice formation–is bypassed. See § 4.1.2 for detailed discussion.

### Appendix B. Dimensionless volume-fraction

The relations below approximate the volume-fraction ratio \( \nu/\nu_0 \), where \( \nu_0 \) is the volume-fraction during the QS regime as shear rate approaches zero. For an arbitrary shear band thickness \( H_\alpha \) and area \( A \), the mass of the sample bulk is

\[ [\text{mass}]_\alpha = \rho \nu_\alpha H_\alpha A \] (B.1)

and \( \rho \) and \( \nu_\alpha \) are material density and volume-fraction. For the same number of particles, the total mass remains constant through any volumetric changes. Then for a different shear band thickness \( H_\beta \),

\[ \nu_\alpha H_\alpha = \nu_\beta H_\beta \quad \text{and} \quad H_\alpha(\dot{\gamma}) = H_\beta + \Delta H(\dot{\gamma}) \] (B.2)

where \( \Delta H(\dot{\gamma}) > 0 \) is the height change–the only measured parameter. The rate-dependent \( H_\beta \) conflicts with our Savage number approximation where shear band thickness is assumed rate-invariant. The inconsistency may be explained by figure 8 where the maximum \( \Delta H(\dot{\gamma}) \) is below the resolution of a single grain–hence \( \Delta H(\dot{\gamma}) \) does not alter our approximate that \( H \approx 2D \). Continuing from (B.2), dimensionless volume-fraction becomes

\[ \frac{\nu_\alpha}{\nu_\beta} = 1 + \frac{\Delta H}{H_\beta} \] (B.3)

Shear band thickness \( H_\beta \) and its corresponding volume-fraction \( \nu_\beta \) are based on the first-order approximation of the shear thickness being \( \approx 2D \) as discussed in § 2. For
clarification, \( \nu = \nu_{\text{static}} \) outside the shear band where grains are statically packed. Using consistent indices by transforming \( \alpha \to 0 \) as the initial QS height, (B3) becomes

\[
\frac{\nu(\dot{\gamma})}{\nu_0} = 1 + \frac{\Delta H(\dot{\gamma})}{2D} > 1.
\] (B4)

**Appendix C. Free-volume ratio**

To compare our model to the experimental data, free-volume ratio \( \phi \) in (5.25) is calculated for constant-pressure experiments. Since volume-fraction is impossible to measure during shear, the measured top-plate height \( H \) is used instead. For a given volume of porous material with unconfined height, the overall volume-fraction is inversely proportional to the column height and a reference height. Here, we choose the absolute maximum volume-fraction and its corresponding column height, explicitly as \( \nu_\infty \) and \( H_1 \) as references in figure 1. Therefore,

\[
\nu = \frac{\nu_\infty H_1}{H} \quad \text{and} \quad \nu' = \frac{\nu'_\infty H_1}{H_2},
\] (C1)

where \( \nu_\infty \) and \( \nu'_\infty \) represent the dynamic volume-fractions as given in the boundary conditions in (5.10a-c). Note, \( \nu'_\infty \) is a state variable and thus non-universal. \( H_1 \) and \( H_2 \) are the associated column heights for each volume-fraction, \( \nu_\infty \) and \( \nu'_\infty \), signifying the dilation process during granular fluidization. The relationship between \( H_1 \), \( H_2 \), and \( H(\dot{\gamma}) \) is

\[
H(\dot{\gamma}) = H_2 + \Delta H(\dot{\gamma}) = H_1 + \delta + \Delta H(\dot{\gamma})
\] (C2)

where \( \delta \) is the 'dilatancy height' that defines the minimum height difference between dynamic and static packing (See figure 1). \( \Delta H(\dot{\gamma}) \) is the measured column height change by the system. For comparison, \( H_1 < H_2 < H_3 \). Therefore, the volume-fraction ratio from (5.25), using (C1), is

\[
\frac{\nu_\infty - \nu'}{\nu_\infty - \nu} = \frac{1 - \frac{H_1}{H_2}}{1 - \frac{H_1}{H}},
\] (C3)

and combine with relation (C2),

\[
\frac{\nu_\infty - \nu'}{\nu_\infty - \nu} = \frac{1 - \frac{H_1}{H_2} + \frac{\delta}{H_2}}{1 - \frac{H(\dot{\gamma})}{H} + \frac{\delta}{H} + \frac{\Delta H(\dot{\gamma})}{H(\dot{\gamma})}}.
\] (C4)

Also assuming \( H_2 \approx H(\dot{\gamma}) \), the interstitial volume ratio \( \phi \) becomes

\[
\phi = \frac{\nu_\infty - \nu'}{\nu_\infty - \nu} \approx \left(1 + \frac{\Delta H(\dot{\gamma})}{\delta}\right)^{-1} \leq 1.
\] (C5)

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