An Experimental Study of the Transitional Regime for Granular Flow: the Role of Compressibility

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Abstract

In this paper we experimentally study the transitional flow of a simple-sheared dry granular assembly. Between the familiar limiting regimes (grain-inertial and quasi-static) of granular flow, the physical description of the intermediate regime remains elusive. Our experiment utilizes a top-rotating torsional shear cell capable of micron accuracy in height and a velocity range of 5 decades ($10^{-3} - 100$ rad/sec). The results show that the shear and normal stresses exhibit an inverse rate-dependence under a controlled-volume environment in the transitional regime while the same experiment captures the limiting regimes in agreement with previous work. The empirical data illustrates a previously unknown ‘dip’ in the stress response to increasing shear rate. Under a controlled-pressure environment, however, the packing fraction is observed to be a positive function of strain rate. We combine the results from both configurations to infer that fluidized granular compressibility, as a function of rate, is a significant factor in granular dynamics. A theoretical model is presented to explain our findings. The formulation provides an equation of state for dynamic granular systems, with state variables of pressure, strain rate, and packing fraction. Lastly, both constant-volume and constant-pressure experiments fit the compressible theory with similar fitting parameters.
1. Introduction

Under continuous strain, granular material exhibits regimes analogous to all three phases of matter, i.e., solid, liquid, and gas (Jaeger et al. 1996). The physical interaction between grains has distinct modes of momentum transfer under different amount of excitations, either from shear, air injection, or vibration. With small excitations, granular material escapes from a jammed state into a mobilized one (Cates et al. 1999; O'Hern et al. 2001; Corwin et al. 2005). Within this regime, the long-duration grain-to-grain interaction is frictional Coulomb-type with momentum exchanging mainly through rubbing and rolling, and interactions often involve many grains in aggregated forms (Savage et al. 1983; Savage 1984). Therefore, slowly deforming granular materials in the quasi-static regime (QS) exhibit a frictional, rate-independent bulk stress response. With increasing excitation, granules can achieve a readily flowing phase. Like in a dense gas, particles within such a flowing regime interact predominantly through short-duration binary collisions (Savage et al. 1983). A modified kinetic theory has been used to describe the underlying physics behind highly agitated granular materials (Savage 1998). Within this dynamic state, commonly referred to as the grain-inertial regime (GI), stresses are quadratically rate-dependent for a highly sheared system. Between the two extreme regimes (GI and QS), however, there exists a transition flow that has yet to be empirically defined.

Studies have formulated both computational and theoretical results on the dynamics of granular flow (Bagnold 1954; Savage 1984; Hanes & Inman 1985; Jop et al. 2006). Other authors have studied granular rheology and the dependencies on shear rate, solid fraction, gravity, and applied pressure (Campbell 1990; Karion & Hunt 1999; Hsiau
& Shieh 2000; Klausner et al. 2000; Tardos et al. 2003; Bossis et al. 2004). Specifically, constitutive equations for shear and normal stresses have been hypothesized for granular hydrodynamics (Savage 1998; Bocquet et al. 2002; Hendy 2005). However, the physics behind the transition between GI and QS regimes have mostly been overlooked, despite theoretical uncertainties (Tuzun et al. 1982; Bocquet et al. 2002) and anomalous empirical data signaling its peculiarity (Tardos et al. 1998; Dalton et al. 2005). Particularly, recent experimental observations have indicated that both contact force distribution (Corwin et al. 2005; Dalton et al. 2005) and geometrical anisotropy (Majmudar & Behringer 2005) evolve within a sheared granular system under intermediate deformation rates. The collective effect of these local flow phenomena on the grain scale escalates the complexities of granular dynamics.

In this study we explore experimental shear rates that span across all three dynamic flow regimes: grain-inertial, transitional, and quasi-static. We then compare results from constant packing fraction and constant confining stress experiments to address granular compressibility. To study the effects of flow anisotropy and force network, two different grain column heights are used. Moreover, a theory is devised based on a velocity-dependent granular compressibility that also includes Coulomb yield conditions and granular dilatancy. From fitting the theory to the experimental data, we find consistent parameters in support of an equation-of-state for non-thermal, non-attractive particle systems. By doing so, we may have captured the experimental ramifications of localized flow phenomenon—the formation and collapse of clusters and force chains—as a self-organized resistance to granular deformation.
2. Important Dimensionless Parameters

Non-cohesive particle flow has a unique set of properties that is analogous to the compressible nature of a dense gas, the visco-elasticity of a colloidal fluid, the plasticity of a common solid, or a combination of all three (Edwards & Mounfield 1996; Savage 1998). These intrinsic properties affect the flow of a granular system differently in various flow regimes. Depending on the frequency and the relative velocities of inter-particle collision, distinctive macroscopic dynamic states can emerge. According to Bagnold’s analysis (1954), the three regimes of a dry, non-cohesive material are grain-inertial (GI), quasi-static (QS), and transitional. These flow regimes distinguish three modes of momentum transport: frictional, collisional, and kinetic (Campbell 2002).

Savage quantified his experiments with the Savage parameter as (Savage 1984)

\[
Sa = \frac{\rho_p D^2 \dot{\gamma}^2}{\sigma}
\]  

(1)

The Savage number is the ratio between the inter-granular collision forces and the consolidation force and it delineates the granular flow regimes. Parameters \(\rho_p\) is the particle density, \(D\) is the mean grain diameter, \(\dot{\gamma}\) is shear rate, and \(\sigma\) is the consolidating stress. Note the consolidation force includes all compacting forces such as gravity or electro-static forces if present.

In the current investigation, we impose an extra assumption that the shear band thickness remains velocity-independent and linearly proportional to the grain diameter. Since the velocity in the flow \(U\) scales with the linear velocity at the wall, \(U_w\), the shear rate \(\dot{\gamma}\) is approximately \(U_w / D\). Substituting in to Eq. (1),

\[
Sa = \frac{\rho_p U^2}{\sigma_w}
\]  

(2)
where $\sigma_w$ is the consolidation stress applied at the wall. The Savage number in Eq. (2) is independent of grain size. It is important to note that this assumption is only valid for boundary sheared flows and not avalanche type flows where shear thickness depends on other factors (Jop et al. 2005).

Recent shear-cell experiments with cylindrical Couette geometries have employed different dimensionless parameters. Tardos et al. (2003) used a dimensionless rate $\dot{\gamma}^* = \dot{\gamma}\sqrt{D/g}$, that incorporates grain diameter $D$ and gravity $g$. Since $\dot{\gamma} \sim U/D$ in our analysis, the dimensionless shear rate becomes,

$$\dot{\gamma}^* = \frac{U}{\sqrt{gD}} \quad (3)$$

Continuing our analysis, the dependence of stress on the system parameters will be different in the various granular flow regimes. Similar to the parameters offered by Savage (1984) with a general functionale for the shear stress in term of other independent variables as

$$\tau_w = fn(\sigma_w, \rho_p, U, g, v', w', v, D, D', e, \mu_f, E, s) \quad (4)$$

where $\sigma_w$ is the applied wall stress, $U$ is translational grain velocity, $v'$ and $w'$ are the fluctuation velocities in the translational and radial directions in the plane of shear, $D$ is the average grain diameter, $D'$ is the sample size variance, or the degree of polydispersity, $g$ is the acceleration of gravity, $e$ is the coefficient of restitution defined as the ratio between incident and reflected velocities, $\nu$ is the bulk solid packing fraction, $\mu_f$ is the inter-grain friction coefficient, $E$ is the bulk elastic modulus of the material and $s$ is the grain aspect ratio which affects the inter-particle friction and
compressibility. Since shear band width is proportional to grain size, the fluidized shear thickness is implicitly included in the analysis.

In Appendix A, the variables in Eq. (4) are non-dimensionalized to determine the important physical parameters within the various granular flow regimes. These dimensionless groups then become the fundamentals of our theory. In our experiments, control parameters are \( \sigma_w, U, \tau_w \) and \( D, \rho_p, g, \mu_f, \) and \( E \) are kept constant.

3. Experiment Setup and Procedure

In the present experiment, we aim to investigate the connection between stress and strain rate for a granular bulk over a wide range of deformation rates (~5 decades). Our objective is to understand the granular flow regimes, with emphasis on the transition regime and its peculiar rheological properties. Using our apparatus, we intend to study the flow phenomenon of particles governed by purely repulsive and frictional interactions. The varying parameters in our experiment are packing density, shear rate, and pressure, while polydispersity, material sample, and grain size are fixed.

To accurately analyze a system of discrete, mobilized elements, many experimental subtleties are implemented to account for the following inherent nature of granular materials: compressibility, jamming threshold (O'Hern et al. 2003), crystallization, and packing configurations. First, the compressible nature of the particle ‘fluid’ warrants two different experimental procedures: constant-volume and constant-pressure (see Section 3.2). In contrast, compressibility is insignificant for the rheology of conventional liquids.
Second, crystallization occurs within systems of equal-sized particles where abrupt changes occur at flow transitions (Drake 1990). To fully explore the granular flow regimes, therefore, the chosen sample must have a large variance in particle size. The polydispersity of an amorphous granular sample may result in a non-monotonic, gradual transition between the GI and QS flowing states. O’Hern et al. (2003) has also suggested that a highly dispersed mix of particles is required to explore the entire granular phase space. Thus, we choose natural beach sand, which is a strongly polydispersed medium, for our experiments. A set of similar experiments on a more uniform material (US Silica 1-Q powder) were dramatically different from the sand experiments and showed no shear-weakening. Similar to glass formation at low temperatures, beach sand is able to sustain its amorphous state at low or zero shear rates.

Third, there exist an infinite number of meta-stable configurations, or stacking, of any given set of particles. These meta-stable packing configurations arise from the frictional nature of interacting grains, where the smallest perturbation in the magnitude or the direction of the compressive stress can disrupt this fragile arrangement (Liu & Nagel 1998). The result of this fragileness is material compaction: the increase in solid volume fraction when the individual particles subject to vibration or deviatoric strain (Duran 1999). Because different packing arrangements do not represent the lowest energy state for the system, particles tend to self-organize into a more stable configuration once enough energy is applied to overcome the friction at the inter-granular contact. Thus in our experiment, a granular sample undergoes a ‘steady-state’ procedure where a continuous strain is applied until compaction does not occur within the experimental time scale of ~hours. With the ‘steady-state’ procedure, the experiment is able to isolate the
changes in packing fraction to a pure effect of shear rate, instead of grain reorganization, during constant-pressure procedures.

3.1 Instrument Setup and Sample Preparation

To investigate the granular flow, we use a top torsional rheometer (AR-2000, TA Instruments Inc.). As seen in figure 2, the system utilizes an upper rotating plate with 20-mm diameter and a lower fixture that can detect normal force (±0.1 N) through an internal force transducer. The rheometer is a highly sensitive, feedback controlled instrument that monitors torque, normal force, gap between the plates, and shear rate simultaneously. For instance, this particular setup exhibits good tribological results for solid-solid contacts using lubricants such as engine oil (Kavehpour & McKinley 2004). Specifically the regions of the lubrication curve (boundary, mixed, and hydrodynamic), together known as the Stribeck curve, have been experimentally identified by using this apparatus.

The disk geometry generates a range of shear rates from zero at the center to a maximum at the rim. As the rheology is strain-rate dependent, stresses likely vary with radius. We can only measure the integrated normal and shear stresses on the plate. These are a weighted average of the radially varying stresses (Kavehpour & McKinley 2004). We assume that the weighting is strain-rate independent and therefore report and model the stresses using a function of circumferential velocity $U_w$ at the wall. Specifically,

$$U_w = \frac{\omega R}{\sqrt{2}} \quad (5)$$

in terms of rotational rate $\omega$ and shear plate radius $R$. 

9
Typically, the system is used to study the rheological behavior of a fluid. In our application, the upper plate is used for compression by controlling normal stress (±1 Pa) or gap (±1 μm), while shearing through a user-specified velocity range. The upper plate (diameter = 20 mm) is prepared by taping 80-grit sandpaper to the face to facilitate stress transmission. Tape is also set on the bottom fixture to prevent slippage. To contain the grain sample, a Teflon self-lubricating sleeve is concentrically aligned around the shearing plate. Although the sidewall friction is unavoidable, the system can be rotationally stress-mapped to counter all of the external resistance by applying a background torque. During optimal alignment, the experiment achieves an average sidewall stress contribution of ≈1% of the total shear stress. An annulus setup with concentric inner and outer walls has been considered but all attempts have failed to reduce wall friction to below 10% of the overall stress.

The samples used in these experiments are beach sand which is composed of grains that are highly rigid and irregular in shape as shown in table 1 and figure 3. The sample is prepared by sieving the sample through a US Std. mesh size 18 and 120, respectively with 1000 μm and 125 μm openings. The sand is made of predominantly quartz and feldspar grains which have a Young’s moduli of 30-70 GPa; thus they are considered highly rigid as compared to our experimental stresses on the order of 10^3 Pa. Other relevant sample properties are listed in Table 1. The solid fraction is measured by weighing a granular column inside of a graduated cylinder and the material is lightly tapped for compaction under gravity.
3.2 Experimental Procedure

The experiment is designed to recover the stress-strain rate relationship. To begin, the granular sample is loaded up to one of two heights: 1 mm and 6 mm that correspond to roughly 3 and 20 grain-diameters, respectively. According to numerical simulations of Tardos et al. (2003) and experimental observations by Hanes and Inman (1985), 10 to 13 grain-diameter is the ‘effective’ shearing zone in a granular layer because of the exponential decay of the velocity with depth. Therefore, based on the published correlation length of 15 to 20 grain diameters, the 3-diameter column will not have force propagating beyond the sheared layer, unlike the 20-diameter column (de Gennes 1999; Majmudar & Behringer 2005). We use the results from the two different column heights to elucidate the role of the highly-correlated contact force anisotropy (Howell et al. 1999).

After loading the granular sample, the granular layer is pre-sheared for 100 rotations to make the packing consistent. Then for every experiment, the sample is sheared from high to low velocities at evenly distributed velocities on a log scale. Results from low to high velocities lack consistency but do have a generally similar appearance. For data acquisition the sampling rate for shear/normal stress and gap is 10 Hz. Other important experimental parameters and their range are given in Table 2. Generally, the velocity range is chosen to stay above the stick-slip regime (Aharonov & Sparks 2004) and below the rates of excessive wear on the system components.

To account for granular compressibility, two general shearing conditions are used: constant-volume and constant-pressure. In the constant-volume procedure, the gap stays fixed while pressure varies with shear rate. The gap height for the specified sample is
system-determined from a pre-experimental procedure of fixed pressure \((10^4 \text{ Pa})\) and velocity \((10 \text{ rad/s})\). Because of volume dilatancy during granular deformation, the gap in the constant-volume procedure must be set while the grains are mobilized. In this configuration, granular solid fraction remains constant.

The second shearing condition is a constant-pressure configuration. Instead of maintaining a constant volume, a user-specified normal stress is feedback-controlled by adjusting the gap height \((\pm 0.1 \text{ micrometer})\). The isotropic pressure remains fixed while packing fraction changes with shear rate. From the loaded height of 6mm, the granular sample is allowed to expand and contract with sub-micron accuracy. The dimensionless packing fraction \(v/v_0\) is calculated from the gap measurements (see Appendix B). The maximum gap changes for the two runs are 47 µm and 58 µm, respectively.

In both constant-volume and constant-pressure procedures, the number of particles stays fixed.

**4. Results and Discussion**

From the experimental results, we want to distinguish the different regimes that are well-established for granular flow. Although the experiment utilizes a torsional Couette geometry, we have analyzed the results below based on a translational and two-dimensional planar geometry. With this idealized assumption, we have neglected any secondary flow effects on the overall granule behavior. But such an assumption is valid only if the secondary flow velocity is much less than the tangential velocities. Hanes and Inman (1985) have used similar assumptions to interpret their torsional shear cell results. Another potentially problematic simplification includes the unknown effect of a wide
distribution of the sample grain size. The Savage number in Eq. (2) neglects polydispersity and describes only the behavior of mono-sized, spherical particles. Therefore, we do not analyze the effects of size variance or grain aspect ratio (jaggedness) on our results. Instead, a statistical mean is introduced for these two factors and a mean diameter is used. However, in hindsight, a largely poly-dispersed medium may induce the flow anisotropy and compressibility otherwise nonexistent in a mono-dispersed medium (Majmudar & Behringer 2005).

4.1 Constant-volume Experiment

To demonstrate the various flow regimes, the constant-volume procedure is first used. As described in Appendices A.1 and A.2, the contact mechanic goes from rate-independent and frictional to rate-dependent and kinetic interactions. In figure 4, the log-log plot of shear stress versus Savage number displays the well-understood GI and QS regimes as well as their transition in between.

As shown in figure 4, the grain-inertial transition occurs around a rotational rate of $\omega = 25\text{rad/sec}$. Converting this rate into $U_w$ using (5), the Savage number in (2) results in $Sa \sim 1$. According to Savage (1984), the GI transition should occur around Savage number of unity. Figure 4 is also closely reminiscent of the powder flow diagram envisioned by Tardos et al. (2003) depicting various granular flow regimes. The dimensionless rate in (3) from Tardos et al. (2003) and Klausner et al. (2000) also results in $\dot{\gamma}_{Gr}^* \sim 1$. The consistency amongst the two common dimensionless rates validates the present rate-dependant stress behavior inferred from our experiment.
4.1.1 Grain-inertial and Quasi-static Flow

In figure 5, the shear stress is plotted against rotation rates through a range of $10^{-3}$ to $10^3$ rad/sec and that corresponds to Savage number on the order of $10^{-11}$ to 1. The two limiting regimes, GI and QS, can be easily identified via visual inspection where the stresses are rate-independent and rate-increasing, respectively. Similar to many experimental and theoretical results showing a quadratic stress dependence on shear rate, our power-law fit also yields a nearly quadratic relationship against shear rate (Savage 1984; Savage & Sayed 1984; Hanes & Inman 1985; Karion & Hunt 1999; Klausner et al. 2000). To fit our results, a power-law is used for shear stress $\tau$ of

$$\tau = \tau_0 + \tau_1 \omega^n \quad \text{or} \quad \tau = \tau_0 + (\tau_1 C) \dot{\gamma}_{wall}^n$$

where $\omega$ is the rotation rate, $\dot{\gamma}_{wall}$ is the shear rate at the wall, and $\tau_0$, $\tau_1$ and $C$ are constants. Eq. (6) is commonly used to describe the Bingham model of a non-Newtonian colloidal fluid above a threshold yield stress for a fluid such as blood (Larson 1999).

From figures 5a and 5b, the rate-dependent shear stress within the grain-inertial regime for poly-dispersed sand is observed. The rate dependence is nearly quadratic where the range of the $n$ value is $1.7 \leq n \leq 2.0$ for shear stress and $1.8 \leq n \leq 2.1$ for normal stress. Previous experimental work done yielded a range of powers, $n$, between 0.75 to 2 in the grain-inertial regime (Savage 1984; Klausner et al. 2000; Sawyer & Tichy 2001; Tardos et al. 2003). The present experiment agrees with this range.

It is also important to point out that all three regimes are revealed for a medium under a single packing fraction. In contrast to previous conclusion drawn by Campbell (2002), it is possible to obtain the GI and the QS regimes from an equally dense medium.
In our case the packing fraction of sand is $\approx 0.61$ and it is close to the maximum random close packing of polydispersed particles of $\approx 0.636$ (Visscher & Bolsterl 1972).

4.1.2 Transitional Flow

In figures 5a and 5b within the transitional regime, shear stress and normal stress are both inversely proportional to shear rate for coarse sand samples. Such inverse stress-rate relation within the transitional regime was also observed by Tardos et al. (1998) in their torque measurements using a Couette cell. In addition, Jaeger et al. (1990) have also deduced a velocity-weakening stress by comparing frictional and gravitation losses during intermediate deformation rates.

One crucial feature arises in the limit of zero consolidating pressure by looking at the trend given in figure 5. The transitional regime, or the ‘dip’ in shear and normal stresses, seems more pronounced for runs with higher normal stresses. Interpreting the above observation, for the limit of zero externally applied stress, the rate-invariant QS regime would extend entirely to the quadratic rate-dependent GI regime. As a result, granular shear flow under low applied pressure may not recover the shear-weakening transition observed here. The above interpretation may explain the missing transitional regime from concentrically arranged shear cells. These geometries utilize gravity as the consolidating pressure on the order of $10^2$ Pa at mid-height for a 10 cm column (Bocquet et al. 2002; Tardos et al. 2003; MiDi 2004).

The exact physics behind shear-weakening of the transitional regime is unknown. However, we offer a qualitative explanation: The observed pressure-induced shear-weakening points to the possible role of force network in sheared granular systems.
shown by Ostojic et al. (2006) that under large loads, the fractal medium adapts to external stresses by aligning particles into filamentary force chains. Moreover, the stress-induced anisotropy also exists geometrically (Majmudar & Behringer 2005). The filamentary chains appear to physically align in the direction of the externally applied deviatoric stress. The above observations can be interpreted as a self-organizational stiffening of granular materials in response to deformation. Due to contact friction, the individual chains within a granular network can resist shear deformation by carrying approximately 10 times the mean external stress (Corwin et al. 2005). As the rate increases, the granular medium loses its spatial anisotropy and therefore its stiffness decreases. The structural breakdown is shown by the shear-weakening behavior in figures 5a and 5b. Also seen in figure 5, under constant packing fraction, the dissimilar stresses within the QS regime seem to converge when approaching the GI regime. The trend indicates the diminishing effect of contact force anisotropy with increasing shear excitations: a consistent interpretation that leads to a more uniform contact force distribution in the GI regime.

For completeness, table 3 shows various dimensionless parameters used to model the stress and frictional behavior of a mobilized dense granular layer. The magnitude defines the approximated dimensionless shear rate for the bifurcation between both GI and transitional, and transitional and QS regimes.

\textit{4.1.3 Frictional behavior}

From using the constant-volume method, we investigate the frictional behavior of the granular layer by comparing the ratio of the shear and normal stresses against Savage
number. Surprisingly, the close resemblance of shear and normal stress behavior produce a friction profile that is independent of shear rate as shown in figure 6. In support of our result according to Savage (1998) and Cheng et al. (1978), both proposed normal stress is proportional to the shear stress, independent of how particles interact.

Figure 6 shows good correlation between 6mm gap and 1mm gap results as the Savage number approaches zero. However, the friction coefficient seems to deviate sporadically at higher velocities near the collisional regime. We believe inconsistencies in the experimental data are due to the emergence of wall slip during high shear rates. The ‘upward’ trend in the friction coefficient indicates less shear stress than expected, and that can be the result of inadequate stress transmission at the wall due to wall slippage.

In the regimes descriptions in Appendix A and the dimensionless groups of (A.2), (A.6), and (A.7), packing fraction is involved in all three regimes of granular flow. In the figure 6, the friction coefficient is compared with the angle of repose \( \theta_{\text{repose}} \) measured from a pile of the same sand sample. The resulting friction coefficient \( \mu \approx 0.78 \), where \( \mu = \tan \theta_{\text{repose}} \), is about 11\% lower than the compressed sand from our shear flow experiment. The above comparison highlights the effect of packing fraction under a controlled gap and rotation rate on a granular medium. A denser medium has less interstitial spaces (increased correlation number) within the granular network. The lack of space inhibits grain mobility, thus producing a higher ratio of shear to normal stress. The analogy is similar to the changes in electron mobility and conductivity when a semiconductor is induced by doping.
4.2 Constant-pressure Experiment

Figure 7 is the semi-log plot of normalized packing fraction vs. Savage number from constant-pressure experiments. Normalized packing fraction inferred from gap measurements (appendix A) spans across the entire range of shear rates. The data indicates conclusively that packing fraction is a function of shear rate under constant pressure.

The results from constant-pressure and constant-volume experiments are closely related. The extremums in applied stresses and packing fraction in figures 5 and 7 occur at the same rotation rate (at $\omega \approx 23$rad/s). Under a non-equilibrated steady-state, the work done at the shearing surface dissipates completely through inelastic particle collisions. The work of surface stresses, given by $\nabla \cdot (\tilde{T} \cdot \tilde{U}_w)$ where $\tilde{T}$ is the total stress tensor, is directly proportional to the applied stress at the wall. Hence in figure 5, the minimum in shear and normal stresses is also the minimum in energy dissipation. Since volume inversely relates to pressure, a local minimum in stress corresponds to a local maximum in packing fraction and vice versa. Thus in figure 7 with pressure kept constant, the maximum in packing fraction occurs at the shearing state which corresponds to the minimum dissipation. The coincidence of extremums suggests that packing fraction, stress, and strain rate are intimately linked through an equation of state that does not allow them to vary independently. Furthermore, the stress minimum and packing fraction maximum indicate an optimum condition for transporting and manipulating granular systems.
4.3 Gap Height Dependence

In figures 8 and 9, dimensionless shear stresses versus rate from the constant-volume experiments for low and high gap configurations (figures 4 and 5) are plotted against Savage number. Normalized normal stresses have similar trends and thus are not shown. In figures 8a and 9a, the shear stress in the QS regime is normalized by the mean normal stress in the rate-independent section. The average friction coefficient is \( \mu_{QS} = 0.88 \) and \( \mu_{QS} = 0.85 \) for 6 mm and 1 mm experimental gaps, respectively.

Figures 8b and 9b offers comparison between high and low gaps for the dimensionless stress in the GI regime. In both plots, shear stresses are normalized by the dynamic granular pressure, \( \rho U^2 \), generated from particle collisions. The dimensionless shear stress data collapsed onto a single curve demonstrates that the Savage number appropriately distinguishes the limiting dynamical regimes.

The similitude illustrated in figure 8 and 9 for both the 6 mm and 1 mm results in the limiting regimes indicates little or no effect from the variable column height. However in the transitional regime, the column height has a significant effect on the overall flow properties as discussed in Section 4.1.2; disparities between the fitting parameters in Section 5.1 support this claim. Therefore, we conclude from dimensionless analysis that the limiting regimes behave consistently regardless of shear thickness, whereas the transitional regime is influenced by the small sample volume.

5. Theoretical Model

A key uncertainty in describing the fluid-like behavior of granular materials is the transition between the collisional and frictional interaction regimes. The inability of the
continuum constitutive relations, such as the Navier-Stokes equation, to account for the heterogeneities of granular flow has been a difficult issue to overcome. To model the macroscopic quantities of granular mediums, such as stress, packing density, or flow profile, many theories rely on the visco-plastic or elasto-plastic description used for metals (Dartevelle 2004). Although such a top-down approach corresponds well with the frictional nature of granular materials by assuming perfect plasticity, recent discoveries indicate otherwise. Specifically, the rate-dependent localized variations in stress and packing density indicate the complexities of granular assembly unaccounted for by the continuum models (Savage 1998).

To capture the macroscopic effect of localized flow, we describe granular flow using analogous concepts from conventional thermodynamics. Similar to pressure, temperature, and density for describing solids and fluids, dry granular flow may be defined by pressure, deviatoric strain rate, and packing fraction. Although granular interactions are fundamentally non-conservative, unlike molecular interactions, the energetically-favorable states are nonetheless achieved under external excitations. The identical velocity at the minimum stress and maximum packing fraction presented in figures 5 and 7 reinforces the notion of thermodynamically-balanced flowing states as discussed in Section 4. Below we develop an equation-of-state that is consistent with our experiments.

To model the physics behind the transitional regime of dense granular flow, our model must satisfy the experimental observations on the packing fraction and shear rate effect in figure 5. Quantitatively, the two effects from our experiments within intermediate shear rates are
(\frac{\partial \sigma}{\partial \nu})_{\dot{\gamma}} > 0 \quad \text{and} \quad (\frac{\partial \sigma}{\partial \dot{\gamma}})_{\nu} < 0 \quad (7)

where \( \sigma \) is the normal stress, \( \nu \) is packing density, and \( \dot{\gamma} \) is shear rate. The first inequality of Eq. (7) is the rigidity relation that describes the compressibility of a granular shear band. The physical explanation of the second inequality in (7) remains elusive.

Changes in packing fraction with respect to shear rate can be derived from the two relations in Eq. (7) using the cyclic rule,

\[
(\frac{\partial \sigma}{\partial \nu})_{\dot{\gamma}} (\frac{\partial \nu}{\partial \dot{\gamma}})_{\sigma} (\frac{\partial \dot{\gamma}}{\partial \sigma})_{\nu} = -1 \quad (8)
\]

and thus at constant pressure,

\[
(\frac{\partial \nu}{\partial \dot{\gamma}})_{\sigma} > 0 \quad (9)
\]

To proceed, we first assume a constant compressibility \( \beta \) for the shear band within a two-phase, mobilized granular system. Under low to moderate pressure, i.e. \( \sigma \ll E \) where \( E \) is the material elastic modulus, the actual volume change of rigid solid constituent is negligible. Therefore, the compressible nature of granular flow is solely due to the volumetric changes in the interstitial spaces amongst the grains. For clarification, the physical meaning of granular compressibility is distinctively different than the irreversible process of granular consolidation (Nedderman 1992; Nowak et al. 1997). For both static and dynamic granular systems, consolidation irreversibly reduces the void fraction. With logarithmically increasing isotropic pressure, packing fraction of sufficiently rigid particles increases until crushing failure of individual grains. To model granular compressibility, we assume the changes in packing fraction are entirely reversible. In this context, the term reversible does not imply an isentropic process as in
classical thermodynamics. Granular interaction is by all means thermodynamically irreversible; energy is always dissipated at the frictional contact. However in our experiments, after the granular medium has consolidated into a steady arrangement, our results show volumetric reversibility as the same medium contracts and expands.

Based on the above considerations, we develop the following theory to fit our experimental data. The theory will explain the results in figures 5 and 7 for GI and QS regimes, along with the transition in between. Up to the GI deformation rates, the compressibility of a mobilized granular solid in terms of the effective interstitial void fraction, \( e \), at constant, non-zero shear rate is

\[
\frac{\partial e}{\partial \sigma_{QT}} = -\beta \quad \text{for} \quad \beta \geq 0
\]

where \( \sigma_{QT} \) is the quasi-static contribution to the isotropic compressive stress. Note, the relation in (10) is only valid within the QS and transitional regimes. The effective interstitial void fraction, \( e \), is defined as the ratio between the expanded ‘free’ volume, which allows for grain mobilization, to the bulk volume at rest (figure 1). It equals zero for a static packing but it has a finite value for a network of moving grains; \( e \) relates to the granular packing fraction \( \nu \) as

\[
\nu = \frac{\nu_{static}}{1+e} \quad \text{and} \quad e = \frac{\nu_{static} - \nu}{\nu}, \quad e \geq 0
\]

and \( \nu_{static} \) is the steady-state packing fraction during static or jammed state, i.e. the densest random close packing. To preserve the uppermost limit of packing fraction for a dynamic pack, however, we replace the maximum static packing fraction \( \nu_{static} \) with \( \nu_{\infty} \).

The effective void fraction is also a measure of grain mobility within a dense packing.
The analogy is similar to the concept of electron/hole concentration in reference electron mobility in semiconductors. Integrating Eq. (10) at constant compressibility and substituting in (11) at a constant shear rate $\dot{\gamma}$, stress yields

$$\sigma_{\text{el}}(\nu) = -\frac{1}{\beta} \left[ \ln(e) + \text{Const} \right] = \frac{1}{\beta} \left[ \ln \left( \frac{\nu}{\nu_{\infty} - \nu} \right) + \text{Const} \right] \tag{12}$$

Separating the integration constant into rate-independent and rate-dependent parts $A$ and $B(\dot{\gamma})$ respectively, Eq. (12) becomes

$$\sigma_{\text{el}}(\nu, \dot{\gamma}) = \frac{1}{\beta} \left[ \ln \left( \frac{\nu}{\nu_{\infty} - \nu} \right) + A + B(\dot{\gamma}) \right] \tag{13}$$

(Note: $\beta \sigma_{\text{el}} \sim 1$ in our experiment so that velocity effect can be important.) Under both constant-volume and constant-pressure conditions, volume fraction depends on both applied stress and shear rate. To solve for the unknowns in (13), we use the inequalities in (7) to formulate the effect of packing fraction on pressure and shear rate. As a result, volume fraction $\nu$ has the following boundary conditions within the QS and the transitional regimes:

$$\begin{cases} 
(1) & \nu = \nu'_{\infty} \quad \text{as} \quad \sigma \to 0, \dot{\gamma} \to 0 \\
(2) & \nu = \nu'_{\infty} \quad \text{as} \quad \sigma \to 0, \dot{\gamma} \to \infty \\
(3) & \nu = \nu_{\infty} \quad \text{as} \quad \sigma \to \infty, \dot{\gamma} \to \infty 
\end{cases} \tag{14}$$

where $\nu'_{\infty}$ and $\nu'_{0}$ are the maximum and minimum packing fractions at the limit of zero applied load, respectively, and $\nu_{\infty}$ is the absolute maximum dynamic packing. However, in actuality for the current experiment, $\nu'_{\infty}$ and $\nu'_{0}$ are the respective absolute maximum and minimum packing fractions for systems under gravity. For comparison, $\nu'_{0} < \nu'_{\infty} \leq \nu_{\infty}$ and all of them are quantities measuring a mobilized particle system. Note,
although there is no absolute minimum packing under zero applied stress, gravity enforces the lower limit on packing fraction in our experiment. Using boundary condition (2) of (14), Eq. (13) becomes

\[
\frac{1}{\beta} \left[ \ln \left( \frac{V'_c}{\nu_c - \nu'_c} \right) + A + B(\dot{\gamma}) \right] = 0 \tag{15}
\]

and if \( B(\dot{\gamma} \to \infty) = 0 \), then \( A = \ln \left( \frac{V'_c}{\nu_c} \right) \). Eq. (15) becomes

\[
\sigma_{QR}(\nu, \dot{\gamma}) = \frac{1}{\beta} \left[ \ln \left( \frac{V'_c}{\nu_c} \right) \left( \frac{V}{\nu_c} \right) + B(\dot{\gamma}) \right] \equiv \frac{1}{\beta} \left[ \ln \left( \frac{V'_c}{\nu_c} \right) + B(\dot{\gamma}) \right] \tag{16}
\]

where the approximation is applicable since packing fraction changes are small that \( \frac{\nu}{\nu_c} \approx 1 \).

We assume that packing fraction is exponentially dependent on strain rate with fitting constants \( C_1 \) and \( C_2 \), while satisfying boundary conditions in (14). Using the relations in (7), the packing fraction is assumed to be a function of pressure and shear rate, namely,

\[
\nu(\sigma, \dot{\gamma}) = \nu_c - \frac{V'_c}{1 - C_1 \exp(-C_2 \dot{\gamma})} \text{func}(\sigma_{QR}) \tag{17}
\]

where \( \text{func}(\sigma_{QR}) \) is an arbitrary stress function to make equation self-consistent with Eq. (16). Note, the rigorous derivation of the exponential function of shear rate is not be included in the current derivation. \( C_1 \) and \( C_2 \) are intended as universal constants for the granular shear flows studied and their physical meaning has not been fully investigated.

Substituting using Eq. (17), Eq. (16) becomes

\[
\frac{V'_c}{\nu_c - \nu} = \frac{1 - C_1 \exp(-C_2 \dot{\gamma})}{\text{func}(\sigma_{QR})} = \exp \left[ \beta \sigma_{QR} - B(\dot{\gamma}) \right] \tag{18}
\]
Solving for \( \text{func}(\sigma_{Qr}) \) and \( B(\dot{\gamma}) \),

\[
\text{func}(\sigma_{Qr}) = \exp[-\beta \sigma_{Qr}]
\]  

(19)

and

\[
\exp[B(\dot{\gamma})] = \frac{1}{1 - C_1 \exp(-C_2 \dot{\gamma})}
\]  

(20)

Then using (19) and (20), Eq. (17) becomes

\[
v(\dot{\gamma}, \sigma_{Qr}) = v_\infty - \frac{v_\infty - v_\infty'}{1 - C_1 \exp(-C_2 \dot{\gamma})} \exp[-\beta \sigma_{Qr}]
\]  

(21)

For completeness, by applying boundary condition (1) of (14), the gravity-induced minimum packing fraction \( v_0' \) is

\[
v(\dot{\gamma} \rightarrow 0, \sigma \rightarrow 0) = v_0' = v_\infty - \frac{v_\infty - v_\infty'}{1 - C_1}
\]  

(22)

and \( \frac{v_\infty'}{v_\infty} \leq C_1 < 1 \) since \( v_\geq 0 \) and \( v \leq v_\infty' < v_\infty \). The magnitude of \( C_1 \) is a function the compacting forces, i.e. external pressure or gravity, at zero shear rate. From Eq. (21), the bulk stresses of a granular solid is

\[
\sigma_{Qr}(v, \dot{\gamma}) = \frac{1}{\beta} \ln \left[ \frac{v_\infty - v_\infty'}{v_\infty - v - \frac{1}{1 - C_1 \exp(-C_2 \dot{\gamma})}} \right]
\]  

(23)

Notice Eq. (23) only accounted for the states within the QS and the transitional regimes. To include the GI regime fit, we looked at the dimensionless groups of relation (A.6) in Appendix A.2. We assume that the stress dependence on shear rate is quadratic. Therefore, similar to the Savage (1998) analysis, the GI stress is

\[
\sigma_{gi} = \text{func} \left( \frac{d'}{\mu_f}, \nu, \epsilon, s \right) \rho D^2 \dot{\gamma}^2 = C_3 \rho D^2 \dot{\gamma}^2
\]  

(24)
The function term is reduced to a fitting constant $C_3$ since the sample and packing fraction are kept constant in the experiment. Combining Eq. (23) and (24), we form a fitting function for our figures 4 and 5, namely,

$$\sigma_{\text{sum}}(\nu, \gamma) = \frac{1}{\beta} \ln \left[ \frac{v_x - v_x'}{v_x - v} \frac{1}{1 - C_1 \exp(-C_2\gamma)} \right] + C_3 \rho D^2 \gamma^2 \quad (25)$$

As for shear stress, by assuming a constant proportionality that $\mu = \tau / \sigma$, it becomes clear that

$$\tau_{\text{sum}}(\nu, \gamma) = \mu \left\{ \frac{1}{\beta} \ln \left[ \frac{v_x - v_x'}{v_x - v} \frac{1}{1 - C_1 \exp(-C_2\gamma)} \right] + C_3 \rho D^2 \gamma^2 \right\} \quad (26)$$

Also, the composite packing fraction for fitting is given as

$$\frac{v_x - v_x'}{v_x - v} = \frac{1}{1 - C_1 \exp(-C_2\gamma)} \exp \left[ \beta \left( \sigma_{\text{sum}} - C_3 \rho D^2 \gamma^2 \right) \right] \quad (27)$$

Constant $C_3$, which is assumed as invariant under constant-volume configurations, is a function of packing fraction $\nu$ according to Eq. (A.6) and the analysis by Savage (1998). The different fit values of $C_3$ in Table 4 support the packing fraction dependence.

Eq. (27) is related to the ratio of the interstitial volumes $e$ and $e_x'$ during particle flow. Specifically,

$$\phi = \frac{e_x'}{e} \approx \frac{v_x - v_x'}{v_x - v} \quad (28)$$

where $\phi$ is the ratio of the interstitial volume $e$ of granular flow against the minimum interstitial volume $e_x'$ as both pressure and shear rate approaches infinity. In Appendix C, experimental $\phi$ is calculated based on measure gap change. Thus, Eq. (27) can be written as
\[
\phi = \left[1 - C_1 \exp(-C_2 \dot{\gamma})\right] \exp\left[\beta\left(\sigma_{\text{sum}} - C_3 \rho D^2 \dot{\gamma}^2\right)\right]
\]  

(29)

5.1 Least-square Fit

In figure 10, the stress versus strain rate data from figure 4 is fit with Eq. (25) for the constant-volume experiment. The log-log plot shows the stress contribution from GI (Eq. (24)) and QS (Eq. (23)) regimes separately, and their sum is in good agreement with the observed flow regimes. In figure 11, the log-log plot presents the comparison between Eq. (29) and the constant-pressure experiment in figure 7. The interstitial volume ratio \( \phi \), as given in Eq. (28), is found using the gap measures with the method described in appendix B. The agreement between hypothesis and empirical data from both gap configurations, along with similar fitting parameters, validates the present theory.

The constants \( \beta \), \( C_1 \), \( C_2 \) are \( C_3 \) are found using an iterative least-square fitting procedure in MATLAB. Table 4 tabulates the fitting constants for experiments of 1 mm and 6 mm gaps as well as constant-volume and constant-pressure configurations. The resulting fit correlate well between 6 mm experiments. The similarity is astonishing when considering the range of both rate-dependent pressure and packing fraction in the respective constant-volume and constant-pressure experiments. In figures 5 and 7, stresses and normalized packing fraction vary as much as one order of magnitude and 15 percent, respectively. However, the coefficients for 1 mm parameters are significantly different than the rest. The disparity indicates the lack of scale-invariance with respect to system size for the fitting parameters \( \beta \), \( C_1 \), \( C_2 \), and \( C_3 \). The shear band for a 1 mm column height contains only 3 grain diameters and that is well below the 10-diameter
shear band in the 6 mm experiments. Thus based on the lack to consistency we speculate that our constants are only consistent for sufficiently large sample volumes.

The a priori value of 13 $\mu$m is used for the dilatancy gap $\delta$ in Eq. (A.16) to calculate the interstitial volume ratio $\phi$ for all fits. Constant $C_1$ in Table 4 is nearly unity. It is restricted to be less than 1 in the fitting procedure as required for non-negative packing fraction as derived in (22). $C_1$ is the minimum packing density for a specific consolidating pressure, i.e. the applied pressure $\sigma$ in our case.

Interestingly, the compressibility of air at standard temperature and pressure (STP) condition is $10^{-5}$ Pa$^{-1}$, about one order of magnitude less than the compressibility of flowing sand in our experiment.

6. Concluding Remarks

In our attempt to recover the transitional granular flow regime, we have seen a glimpse of how particles collectively respond to simple shear. The five decades of shear rate in the present experiment correlates many natural and industrial processes: avalanche, land slide, dredging, and particle fluidization for mixing, segregation, and compaction. Our results show:

1. During constant packing fraction, a shear-weakening behavior is obtained during transitional flow rates. Both shear and normal stresses ‘dip’ and reaches a minimum value.

2. During constant pressure, shear-compaction occurs during the same velocity range as the shear-weakening phenomenon. The mechanism is a reversible
process; packing fraction plateaus and then decreases at lower and higher shear rates during a single experiment.

3. The transitional flow regime is much broader than previous observations spanning a Savage number range from $10^{-7}$ to $10^{-1}$.

4. An equation-of-state devised based on a rate-dependent granular compressibility unifies our findings. Fitting parameters remain consistent for experimental data sets with adequate sample size.

The present theory provides insight into the compressibility of non-cohesive particle systems during steady-state deformation. Independent variables are pressure, shear rate, and packing fraction—or precisely the effective void fraction. Yet the theory has its shortcomings. In particular, it does not consider material properties such as the coefficient of restitution, polydispersity, and grain aspect ratio. We do, however, suspect inelastic collisions, as well as boundary effect, to be a part of a separate constitutive law that balances the inter-granular momentum and energy (Jop et al. 2005). A law of motion that resolves velocity profile, granular temperature, dissipation, and convection (Knight et al. 1996).

Despite being an incomplete description of granular flow, the current theory establishes the various granular flow regimes. Moreover, the rate-dependent compressible theory formulated for the transitional regime may reflect the emerging macroscopic effect of the self-organizational network of force chains. Although the local contact forces and packing fraction are highly probabilistic (Behringer et al. 1999), the ensemble spatial and
temporal averaging of the physical quantities provides a deterministic granulodynamics as indicated by our theory and experiment.

To achieve universality, our model in Eq. (25) and Eq. (29) must also predict the intensive properties for granular flow in other configurations. Certainly, an extension is needed to account for disparate methods of excitation. The shear-driven deformation in our case is unlike gas- or vibration-induced fluidization. It will be interesting to see a comparison with the results from a chute flow or fluidized bed. The comparison may further elucidate the exact physical meanings of the fitting constants $C_1, C_2, C_3$ used in our equations. At this time, the parameters for our theory are only tested within the premise of our experimental setting. The theory has not considered grains with a different size distribution, shape asperity, or rigidity.

Our future work also includes a new annulus setup. A confined channel will help to eliminate any possible effect biased to the shear rate variation of our current setup. Material confinement and excessive wear are the difficulties to overcome for the current annulus geometry. Preliminary results show good agreement with the present flat-plate configuration.
Appendix A.

A.1 Quasi-Static Regime

From shear-dilation and overcoming a jammed configuration, dry granular flow is classified as quasi-static—a regime characterized by a combination of vanishing shear rate, high particle solid fraction and rigidity, and relatively high stress levels (Savage 1984). The granular layer during quasi-static deformation can be characterized as a rigid and interlocking solid layer, along with uneven distribution of contact forces as the Savage number approaches zero.

To understand the pertinent parameters within the QS regimes, we choose applied stress $\sigma_w$, particle density, $\rho_p$, and gravity, $g$ to quantify the regime. Thus Eq. (4) in terms of dimensionless groups for the quasi-static regime is

$$
\frac{\tau_w}{\sigma_w} = f_n\left(\frac{U\rho^{1/2}}{\sigma_w^{1/2}}, \frac{\nu'\rho^{1/2}}{\sigma_w^{1/2}}, \frac{w'\sigma_w^{1/2}}{\rho^{1/2}g}, \frac{\rho g D}{\sigma_w}, \frac{\rho g D'}{\sigma_w}, \frac{E}{\sigma_w}, \mu_f, \nu, e, \sigma \right)
$$

(A.1)

In our analysis for large applied loads, the gravitational body forces can be neglected in relation (A.1). The particle Young’s modulus, $E$, is also assumed to be sufficiently high compared to the applied stress, $\sigma_w$, that the particles are rigid. From the above assumptions, for vanishing shear velocity and thus neglecting inelastic collision effects, relation (A.1) becomes

$$
\frac{\tau_w}{\sigma_w} = f_n(\mu_f, \nu, s)
$$

(A.2)

The resulting stress behavior is mainly a function of particle friction and packing fraction. The overall macroscopic properties are also rate-independent for a granular layer operating in the quasi-static regime. The grain aspect ratio affects to the packing fraction.
A.2 Grain-inertial Regime

In a rapid shear flow, instead of a frictional sliding contact between grains, it is the inelastic collisions that are responsible for most of the momentum transport. The solid concentration is low and particles have a random fluctuation velocity component in addition to the mean velocity (Savage 1984). For the GI regime, we choose particle density $\rho_p$, grain diameter $d$, and shear velocity $U$ for the repeating variables. Thus Eq. (4) in its dimensionless groups for the Savage number greater than unity becomes

$$\frac{\tau_w}{\rho_p U^2} = f(n) \left( \frac{\sigma_w}{\rho_p U^2}, \frac{v'}{U}, \frac{w'd'}{U}, \frac{d'}{\rho_p U^2}, \frac{E}{U^2}, \mu_f, \nu, \epsilon, s \right)$$  \hspace{1cm} (A.3)

For highly rigid, frictional particles under high levels of stress, the pressure and traction at the wall will only depend on the highest order in shear velocity, $U$. The r.m.s. velocity fluctuation $v'$ and $w'$ during slow deformation, also known as the square root of granular temperature, was shown by Loser et al. (2000) and Bocquet et al. (2002) to follow a power-law dependence on shear rate,

$$v' \sim \dot{\gamma}^{0.4}$$  \hspace{1cm} (A.4)

From the assumption that $\dot{\gamma} \sim U / D$,

$$v' \sim U^{0.4}$$  \hspace{1cm} (A.5)

is also satisfied. Therefore from the relation(A.4), it is fair to neglect the translational velocity fluctuations $v'$ and $w'$ for relatively high shear rate/velocity.
Analogous to the procedure done in the previous section, the particle-kinetic quality of granular material is revealed by neglecting gravitational and lower order velocity terms in (A.3). The resulting mean shear and normal stresses, $\tau$ and $\sigma$, are described as

$$\frac{\tau_w}{\rho_p U^2} = fn\left(\frac{\sigma_w}{\rho_p U^2}, \frac{d'}{d}, \mu_f, v, e, s\right)$$  \hspace{0.5cm} (A.6)

Stresses are proportional to the square of the r.m.s. velocity, or the granular kinetic energy. Equation (A.6) is comparable to the relations given in Bagnold’s analysis (1954).

A.3 Transitional Flow Regime

In the limiting flow regimes discussed above, bulk stresses are controlled by different mechanisms and physical parameters as mentioned in Appendices A.1 and A.2. Therefore, the transitional regime between them should consequently exhibit features that resemble both rate-dependent and rate-independent regimes. Depending on the relative stress levels and packing density and assuming the compressible nature of granular packing density, various dimensionless groups in (A.2) and (A.6) can be significant. We anticipate the importance of first order velocity and fluctuation terms in addition to the quasi-static groups. Hence,

$$\frac{\tau_w}{\sigma_w} = fn\left(\frac{U \rho^{1/2}}{\rho_p \sigma_w^{1/2}}, \frac{v' \rho^{1/2}}{\sigma_w^{1/2}}, \frac{w' \sigma_w^{1/2}}{\rho^{1/2} g}, \frac{v}{U}, \frac{w'd'}{d}, \mu_f, v, e, s\right)$$  \hspace{0.5cm} (A.7)
Appendix B.

For calculating packing fraction ratio $\nu / \nu_0$ where $\nu_0$ is the packing fraction during QS regime as shear rate approaches zero. For a given column with height $H_\alpha$ and area $A$, the mass of the grain sample is

$$[\text{mass}]_a = \rho \nu_a H_\alpha A \quad (A.8)$$

and $\rho$ and $\nu_a$ are material density and packing fraction. For the same number of particles, the total mass remains constant through any volumetric changes. For a different column height $H_\beta$,

$$\nu_\alpha H_\alpha = \nu_\beta H_\beta \quad (A.9)$$

From the measures gap change, $\Delta H$, $H_\beta = H_\alpha + \Delta H$. Therefore, Eq. (A.9) becomes

$$\frac{\nu_\alpha}{\nu_\beta} = 1 + \frac{\Delta H}{H_\alpha} \quad (A.10)$$

Shear band thickness $H_\alpha$ and its corresponding packing fraction $\nu_\alpha$ are based on the first-order approximation of the thickness being $D$ as discussed in the text. We also postulate that the packing fraction $\nu$ is uniform within $H_\alpha$ and $\nu = \nu_{\text{static}}$ outside of $H_\alpha$. Thus, $H_\alpha \approx D$ where $D$ is the mean grain diameter. Using consistent indices, and transforming $\alpha \to \sigma$, Eq. (A.10) becomes

$$\frac{\nu(\dot{\gamma})}{\nu_0} = \left[ \frac{\nu_\alpha}{\nu(\dot{\gamma})} \right]^{-1} = \left[ 1 + \frac{\Delta H(\dot{\gamma})}{D} \right]^{-1} \quad (A.11)$$
Appendix C.

To compare the theory to the experimental data, for the constant-pressure experiments, Eq. (27) is used. For constant-pressure experiments, the packing fraction $\nu(\dot{\gamma})$ is only a function of rate. Therefore from using the experimental gap $H$ measurements, $\nu(\dot{\gamma})$ is found by the analysis below.

For a given volume of porous material with unconfined height, the overall packing fraction is inversely proportional to the column height and a reference point. Here, we choose the absolute maximum packing fraction and its corresponding column height, explicitly as $\nu_\infty$ and $H_1$ (Figure 1). Therefore,

$$\nu = \frac{\nu_\infty H_1}{H} \quad \text{and} \quad \nu'_\infty = \frac{\nu_\infty H_1}{H_2}$$  \hspace{1cm} (A.12)

Where $\nu_\infty$ and $\nu'_\infty$ represent the dynamic packing fractions as given in the boundary conditions in (14). Note, $\nu'_\infty$ is a state variable and thus non-universal. $H_1$ and $H_2$ are the associated gap heights for each packing fraction, $\nu_\infty$ and $\nu'_\infty$, signifying the dilation process during granular fluidization. The relationship between $H_1$, $H_2$, and $H(\dot{\gamma})$ is

$$H(\dot{\gamma}) = H_2 + \Delta H(\dot{\gamma}) = H_1 + \delta + \Delta H(\dot{\gamma})$$  \hspace{1cm} (A.13)

where $\delta$ is the ‘dilantancy gap’ that defines the minimum gap difference between dynamic and static packing (See figure 1). $\Delta H(\dot{\gamma})$ is the measured gap change by the system. For comparison, $H_1 < H_2 < H$. Therefore, the packing fraction ratio from Eq.(27), using Eq. (A.12) is
\[
\frac{v_\infty - v'_\infty}{v_\infty - v} = 1 - \frac{H_1}{H_2}
\]
\[
\frac{v_\infty - v'_\infty}{v_\infty - v} = 1 - \frac{H_1}{H_2}
\]

and combine with relation (A.13),

\[
\frac{v_\infty - v'_\infty}{v_\infty - v} = 1 - \frac{H_2}{H_2} + \frac{\delta}{H_2} + \frac{\Delta H(\dot{\gamma})}{H(\dot{\gamma})}
\]

and assuming \(H_2 \approx H(\dot{\gamma})\), the interstitial volume ratio \(\phi\) becomes

\[
\phi = \frac{v_\infty - v'_\infty}{v_\infty - v} \approx 1 + \left(1 + \frac{\Delta H(\dot{\gamma})}{\delta}\right)^{-1}
\]

REFERENCES


DRAKE, T. G. 1990 Structural features in granular flows. *95*, 8681-8696.


<table>
<thead>
<tr>
<th>Properties</th>
<th>Beach Sand</th>
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<tbody>
<tr>
<td>Material</td>
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<td>Density, $\rho$</td>
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**TABLE 1. Sample properties**
### System Parameters

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<th>Constant-volume</th>
<th>Constant-pressure</th>
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<tbody>
<tr>
<td><strong>Min Rotational Velocity</strong>&lt;br&gt;(translational)</td>
<td>0.001 rad/s ($7.1 \times 10^{-6}$ m/s)</td>
<td>0.001 rad/s ($7.1 \times 10^{-6}$ m/s)</td>
</tr>
<tr>
<td><strong>Max Rotational Velocity</strong>&lt;br&gt;(translational)</td>
<td>100 rad/s (0.71 m/s)</td>
<td>100 rad/s (0.71 m/s)</td>
</tr>
<tr>
<td><strong>High / Low Column ($\Delta H$)</strong></td>
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<td>6mm ($\approx 0.050$mm)</td>
</tr>
<tr>
<td><strong>Normal Stress ($\Delta \sigma$)</strong></td>
<td>$\approx$10 kPa ($\approx$10 kPa)</td>
<td>7000 Pa</td>
</tr>
<tr>
<td><strong>Shear Stress ($\Delta \tau$)</strong></td>
<td>$\approx$10 kPa ($\approx$8 kPa)</td>
<td>6800 Pa</td>
</tr>
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</table>

**TABLE 2.** System Controlled Parameters. Minimum and maximum velocities are the applied velocity range. The equivalent translational velocity is given by $\left(\frac{1}{\sqrt{2}}\right)\omega R$, for $\omega$ and $R$ are the rotational velocity and the radius at the rim. The gap range is determined controlled by the amount of material. The column height is user-specified during constant-volume procedures. The normal and shear stresses are user-specified during constant-pressure procedures. Symbol $\Delta H$, $\Delta \sigma$ and $\Delta \tau$ represents the maximum change observed in each parameter within the experiment from all of the runs under investigation.
<table>
<thead>
<tr>
<th>Flow Regimes</th>
<th>$\frac{\rho_p U^2}{\sigma_w}$</th>
<th>$\frac{U}{\sqrt{gD}}$</th>
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<td>&lt;= 10^{-3}</td>
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<td>In between</td>
<td>In between</td>
</tr>
<tr>
<td>Grain-inertial</td>
<td>&gt; 0.1</td>
<td>&gt; 1</td>
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TABLE 3. Summary of various granular flow regimes with corresponding dimensionless parameters for coarse sand grains (mean diameter = 0.3 mm).
<table>
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<tr>
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<th>(CG_{1\text{mm}})</th>
<th>(CG_{6\text{mm}})</th>
<th>(CF_{7\text{kPa}})</th>
<th>(CG/CF)</th>
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<td>(\beta \times 10^4) [Pa(^{-1})]</td>
<td>15 ± 4</td>
<td>6.1 ± 0.4</td>
<td>5.9 ± 0.3</td>
<td>1.0</td>
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<td>(C_1)</td>
<td>≈1</td>
<td>≈1</td>
<td>≈1</td>
<td>1.0</td>
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<tr>
<td>(C_2 \times 10^5) [sec]</td>
<td>19 ± 6</td>
<td>2.1 ± 0.8</td>
<td>2.5 ± 0.6</td>
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<tr>
<td>(C_3 \times 10^3)</td>
<td>13 ± 6</td>
<td>0.9 ± 0.7</td>
<td>2.1 ± 0.5</td>
<td>0.42</td>
</tr>
<tr>
<td>(\delta \times 10^6) [m]*</td>
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<td>13</td>
<td>13</td>
<td>1</td>
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</table>

TABLE 4. Summary of the constants used for fitting. The values are the average value amongst all experimental data. The CG/CF ratio compares the parameters fit to the 6 mm constant-volume data to those from the constant-pressure data. A ratio of unity signifies the consistency between the fitting parameters across different experiments. The ± values represent standard deviations of the parameters for the best-fit to each of the experimental runs under the given conditions. \(C_1\) is approximately unity but it must satisfy the constraint of \(C_1 < 1\) for materials under consolidation. The values for 1 mm experiments are significantly different than the values for other experiments (See text). Averaging involves 5 runs each for 1mm and 6mm constant-volume experiments and 2 runs for constant-pressure experiment. *The same value for the dilatancy volume, \(\delta\), is imposed for all experiments a priori to solve for \(\phi\) in Appendix C.
FIGURE 1. Schematic of static and dynamic regimes of granular flow. a) The static volume of column height $H_1$. Shaded and unshaded regions indicate volumetric void and solid fractions, respectively. b) Shaded (orange) region shows the ‘free’ volume expanded from static packing. The total column is $H_2$ and the dilation constant is $\delta$. The exponential velocity profile has been observed previously (Karion & Hunt 2000). The effective void fraction $\phi$ is the ratio between the dynamic free expanding volume and the original static volume. Diagram is not drawn to scale.
FIGURE 2. Illustration of torsional shear cell. (a) Schematic showing the experimental setup using a cylindrical shear cell with a radius of 10cm and a polished acrylic jacket for lateral confinement. b) Rheometer TA Instruments AR-2000 c) Actual photo of the polydisperse beach sand used as the sample have a mean diameter \( d \approx 300 \ \mu m \) (95% of grains have \( d < 450 \ \mu m \)). The experimental setup has a column height \( h \approx 6.5 \ mm \) and the top torsional plate exerts pressure \( \sigma \) and shear stress \( \tau \) through a roughened surface contacting the grains with an order of magnitude of \( 10^4 \ Pa \). The granular medium has a random packing solid volume fraction \( \nu \approx 0.59 \). Experiment measures, record, and controls \( \tau, \sigma, H, \) and rotation rate from the top plate boundary. Angular velocity ranges from \( 10^{-3} \) to \( 10^2 \) rad/sec that corresponds to \( 10^{-5} \) to 1 m/sec at the rim.
FIGURE 3. Micrographs of actual grains using KP-D50 Digital optical microscope (Hitachi) under 20X magnification.
FIGURE 4. Log-log plot of shear stress vs. Savage number. Both experiments involve a small column height of around 2—3 times the grain diameter. Savage number is calculated at the transition into both grain-inertial and quasi-static regimes to be $Sa=1$ and $Sa=10^{-7}$, respectively.
FIGURE 5. Log-log plot of wall stresses vs. rotational rate for a densely packed beach sand medium. Numbers indicate measured column height for each run. Two runs have lower applied pressure than the other three. 

\( a) \) Log-log plot of shear stress \( \tau \) as a function of angular velocity. 

\( b) \) Log-log plot of normal stress \( \sigma \) as a function of angular velocity. 

The insets of both \( a \) and \( b \) shows the power-law correlation fitted to the data along with the coefficient \( n \) of the fitting equation \( \tau = \tau_0 + (\tau_1 C) \dot{\gamma}_{\text{wall}}^n \). Fitting curves are not linear because of the constant \( \tau_0 \). Both plots also include the granular flow regimes that represent the dominant particle interaction: frictional and collisional. Various flow regimes are classified as: Quasi-static regime where the stress obeys velocity-independent behavior, transition/mixed regime where stress obeys velocity-weakening behavior, and grain-inertial regime where stress obeys Bagnold scaling of \( \dot{\gamma}^2 \).
FIGURE 6. Semi-log plot showing coefficient of friction vs. Savage number for beach sand grains (D = 0.3 mm) by using the constant-volume method. The plot compares experimental data from high and low shear column heights. The higher gap runs exceed the shear band thickness of \( \approx 10 \) grain diameters. The lower gap has 3 grain diameters across. \( \theta \) is the angle of repose measured from the slope of a static granular pile. The observed increase of the friction coefficient (tan \( \theta \)) of all runs compared to the angle of repose of a static pile may indicate the dependence of friction coefficient on packing fraction. Under external pressure, as compared to gravity, grains have less mobility due to higher packing fraction, resulting in a higher friction coefficient.
FIGURE 7. Semi-log plot of normalized packing fraction vs. Savage number. Top axis indicates the corresponding rotational rate $\omega$ for the given Savage number $Sa$. Normal stress is maintained at $7 \pm 0.2$ kPa. Experimental grain column is approximately 6 mm. The dimensionless packing fraction $V/V_0$ is calculated from the gap measurements (see Appendix A). The maximum gap change for run 1 and run 2 are 47 µm and 58 µm, respectively.
FIGURE 8. High Gap (≈6mm) dimensionless stress vs. Savage number. The insets are the stress vs. rotation rate plots as shown in fig. 3. a) Semi-log plot of normalized QS shear stress vs. Savage number. The normal stress $\sigma_{QS}$ used is the averaged normal stress in the rate-independent QS regime. The mean friction of $\mu = 0.85$ shows good agreement among different measurements, showing the rate-independent QS stress behavior. b) Linear plot of normalized shear stress vs. Savage number. The dynamic pressure $\rho U^2$ used is the instantaneous value at each data point. The dimensionless parameter $\tau / \rho U^2$ approaches $\approx 1$ as the Savage number approaches $\approx 1$, showing the granular kinetic effect on stress in the GI regime.
FIGURE 9. Low Gap (≈1mm) dimensionless stress vs. Savage number. The insets are the stress vs. rotation rate plots as shown in fig. 2 (with 3 additional runs). a) Semi-log plot of normalized QS shear stress vs. Savage number. The normal stress $\sigma_{qs}$ used is the averaged normal stress in the rate-independent QS regime. The mean friction of $\mu = 0.88$ shows good agreement among different measurements, showing the rate-independent QS stress behavior. b) Linear plot of normalized GI shear stress vs. Savage number. The dynamic pressure $\rho U^2$ used is the instantaneous value at each data point. The dimensionless parameter $\tau / \rho U^2$ approaches $\approx 1$ as the Savage number approaches $\approx 1$, showing the granular kinetic effect on stress in the GI regime.
Figure 10. Log-log plot comparing data and theory for pressure vs. shear rate from the constant-volume experiments. Grain columns are 1mm and 6mm high. The plot shows experimental data of pressure vs. shear rate. The fit has two sets for 1mm and 6mm experiments. The set includes: Blue point-dash line uses the QS description of granular flow with the addition of the exponential rate contribution (Eq. 25). Green dash line uses the GI description of granular flow (Eq. 26) and has quadratic rate dependence. Red solid line is the sum of blue and green lines (Eq. 27). The fit represents a theoretical equation-of-state for a sheared granular system. The fitting coefficients $\beta$, $C_1$, $C_2$, and $C_3$ are similar to the coefficients fit to the constant-pressure data in figure 11 and they are summarized in Table 4. Other results totaling 5 runs each from 6mm and 1 mm experiments shows very similar fit and fitting parameters.
Figure 11. Log-log plot of comparison between theory and effective void ratio $\phi$ vs. shear rate for the constant-pressure experiment. Grain column is 6 mm. The fitting set includes: Blue point-dash line uses the QS description of granular flow with the addition of the exponential rate contribution. Green dash line uses the GI description of granular flow and it is quadratic rate dependent. Red solid line is the product of blue and green lines (Eq. 29). The fit represents a theoretical equation-of-state for a sheared granular system. The values of fitting coefficients $\beta$, $C_1$, $C_2$, and $C_3$ are similar to the coefficients fit to the constant-volume data in figure 10 and they are summarized in Table 4. One other constant-pressure run shows very similar fit and fitting parameters.