Solutions of the wave equation as superpositions of its localized solutions

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Abstract of the talk

I deal with the wave equation with constant coefficient $c$:

$$u_{tt} - c^2(u_{xx} + u_{yy}) = 0, \quad c = \text{const.}$$
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First, an exact solution [Kiselev, Perel, 1998,2000] will be discussed. Both the solution and its Fourier transform are given with simple explicit formulas, depending on parameters. For some parameters it represents a wave packet with the Gaussian envelop moving along a straight line with the speed $c$. Its asymptotics will be given.

Well-known Fourier analysis enables one to decompose a solution as an integral superposition of plane waves. An analogue of the Fourier analysis in our considerations will be continuous wavelet analysis.
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Then, I show that any solution may be decomposed as an exact integral superposition of the elementary localized solutions obtained from one solution by some transformations. Well-known Fourier analysis enables one to decompose a solution as an integral superposition of plane waves. An analogue of the Fourier analysis in our considerations will be continuous wavelet analysis.
Explicit formula for packet-like solution in 2D moving in $z$ direction:

$$
\Psi(r, t) = \frac{1}{\sqrt{z + ct - i\varepsilon}} \exp \left[-p \sqrt{1 - \frac{i\theta(r, t)}{\gamma}}\right],
$$

$$
\theta(r, t) = z - ct + \frac{x^2}{z + ct - i\varepsilon},
$$

where $p$, $\varepsilon$, and $\gamma$ are free positive parameters.
Exact solution by Kiselev and Perel (2000)

Explicit formula for packet-like solution in 3D moving in $z$ direction:

$$\Psi(r, t) = \frac{1}{z + ct - i\varepsilon} \exp \left[ -p \sqrt{1 - \frac{i\theta(r, t)}{\gamma}} \right],$$

$$\theta(r, t) = z - ct + \frac{x^2 + y^2}{z + ct - i\varepsilon},$$

where $p$, $\varepsilon$, and $\gamma$ are free positive parameters.
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where $p$, $\varepsilon$, and $\gamma$ are free positive parameters.

If $p$ is small, the wave is localized near a surface $r = ct$:

$p = q = 0.5, q = p\varepsilon/(4\gamma), t = 0, t = 5.$
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where $p$, $\varepsilon$, and $\gamma$ are free positive parameters.

If $p$ is small, the wave is localized near a surface $r = ct$: $p = q = 0.5$, $t = 5$, $t = 10$. 

![Graphs showing the wave function for different time points](image1.png)
Exact solution by Kiselev and Perel (2000)

Explicit formula for packet-like solution in 2D moving in $z$ direction:

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where $p$, $\varepsilon$, and $\gamma$ are free positive parameters.

If $p \gg 1$, it behaves as a particle:

$p = q = 10, q = p\varepsilon/(4\gamma), \ t = 0, \ t = 10.$
Exact solution by Kiselev and Perel (2000)

Explicit formula for packet-like solution in 2D moving in $z$ direction:

$$\Psi(\mathbf{r}, t) = \frac{1}{\sqrt{z + ct - i\varepsilon}} \exp \left[ -p \sqrt{1 - \frac{i\theta(\mathbf{r}, t)}{\gamma}} \right],$$

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where $p$, $\varepsilon$, and $\gamma$ are free positive parameters.

If $p \gg 1$, it behaves as a particle: $p = q = 10$, $t = 10$, $t = 30$. 

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Asymptotic simplifications, $p \to \infty$, moderate time

Expand square root in the exponent in the Taylor series

\[-p\sqrt{1 - i\frac{\theta}{\gamma}} = -p\left(1 - i\frac{\theta}{2\gamma} + \frac{\theta^2}{8\gamma^2} + \ldots\right)\]

and take into account that

\[\theta(r, t) = z - ct + \frac{x^2}{z + ct - i\varepsilon},\]
Asymptotic simplifications, $p \to \infty$, moderate time

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$$-p \sqrt{1 - i \frac{\theta}{\gamma}} = -p(1 - i \frac{\theta}{2\gamma} + \frac{\theta^2}{8\gamma^2} + \ldots)$$

and take into account that

$$\theta(r, t) = z - ct + \frac{x^2}{z + ct - i\varepsilon},$$

$$\Psi(r, t) \sim \exp \left( i\kappa(z - ct) - \frac{(z - ct)^2}{2\sigma^2_\parallel} - \frac{x^2}{2\sigma^2_\perp} \right),$$

where $\kappa = \frac{p}{2\gamma}$ is a wave number, $\kappa\sigma_\parallel = \sqrt{p}$, $\kappa\sigma_\perp = \sqrt{q}$, $q = \frac{\kappa\varepsilon}{2}$. 
Asymptotic simplifications, $p \to \infty$, moderate time

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\Psi(r, t) \sim \exp \left( i \kappa (z - ct) - \frac{(z - ct)^2}{2\sigma_{\parallel}^2} - \frac{x^2}{2\sigma_{\perp}^2} \right),
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$p = 100, q = 100$

$p = 9, q = 9$
Asymptotic simplifications, $p \to \infty$, moderate time

$$\Psi(r, t) \sim \exp \left( \frac{i \kappa (z - ct)}{2 \sigma^2_{\parallel}} - \frac{(z - ct)^2}{2 \sigma^2_{\perp}} - \frac{x^2}{2 \sigma^2_{\perp}} \right),$$

where $\kappa = \frac{p}{2 \gamma}$ is a wave number, $\kappa \sigma_{\parallel} = \sqrt{p}$, $\kappa \sigma_{\perp} = \sqrt{q}$, $q = \frac{\kappa \epsilon}{2}$.

$p = 100, q = 9$

$p = 9, q = 100$
Asymptotic simplifications of the solution

\( p \to \infty \), large-time behavior

\[
\Psi(r, t) \sim \frac{1}{2R} \exp \left( i\kappa (R - ct) - \frac{(R - ct)^2}{2\sigma^2_{||}} - \frac{\chi^2}{2\sigma^2_{\chi}} \right), \quad R = \sqrt{x^2 + y^2}.
\]

FIG. 2. \( \Psi \) on the plane \((y, z)\) for \( x=0 \) in successive moments of time from \( t=-2 \) to \( t=6 \) in conventional units. The stronger field is shadowed darker. Dashed lines indicate the cone inside which the packet propagates. (a) \( \kappa d = 6\pi \), (b) \( \kappa d = 18\pi \),

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Asymptotic simplifications of the solution

\[ p \to \infty, \text{ large-time behavior} \]

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\]
The Fourier transform of the solution has been calculated exactly

$$\hat{\Psi}(k, t) = C \exp \left[ - \frac{(k + k_z) \gamma}{2} - \frac{(k - k_z) \xi}{2} - \frac{p^2}{2\gamma(k + k_z)} - ikct \right] \frac{k(k + k_z)}{k(k + k_z)},$$

where $k = |k|$, $C = const$. This formula shows that $\hat{\Psi}(k)|_{k=0} = 0$ and the solution has all zero moments in spatial coordinates. $p = 0.5, q = 0.5$

$p = 100, q = 100$
The Fourier transform of the solution has been calculated exactly

\[ \hat{\Psi}(k, t) = \frac{C \exp \left[ -(k + k_z)\frac{\gamma}{2} - (k - k_z)\frac{\xi}{2} - \frac{p^2}{2\gamma(k+k_z)} - ikct \right]}{k(k + k_z)} \]

where \( k = |k|, \quad C = \text{const.} \). This formula shows that \( \hat{\Psi}(k)|_{k=0} = 0 \) and the solution has all zero moments in spatial coordinates. \( p = 100, q = 9 \) \( p = 9, q = 100 \)
How we obtained this solution

We seeked solution in the class of "relatively undistorted waves" of Courant and Hilbert:

$$u(r, t) = f(\theta)g(r, t).$$

We demanded that $u$ satisfies the wave equation for an arbitrary $f$. This condition yields:

$$\theta^2_t = c^2(\theta^2_x + \theta^2_y),$$

$$2(\theta_t g_t - c^2(\nabla\theta, \nabla g)) + c^2 g \Box \theta = 0,$$

$$\Box g = 0.$$

Bateman and Hillion suggested to put

$$\theta = z - ct + \frac{x^2}{z + ct - i\varepsilon},$$

$$g = (z + ct - i\varepsilon)^{-1}.$$

Kiselev and Perel suggested a choice

$$f = \exp(-p\sqrt{1 - i\theta/\gamma}).$$
I will consider two problems for the wave equation.

1) initial-value problem

\[ u_{tt} - c^2(u_{xx} + u_{yy}) = F(x, y, t), \]
\[ u|_{t=0} = 0, \quad u_t|_{t=0} = 0; \]

2) initial-boundary value problem in a half-plane

\(-\infty < x, < \infty, \ z \geq 0,\)

\[ u_{tt} - c^2(u_{xx} + u_{yy}) = 0, \]
\[ u(x, 0, t) = f(x, t) \]
\[ u(x, 0) = 0, \quad u_t(x, 0) = 0, \]

where \( f(x, t) = 0 \) when \( t \leq 0. \) The aim is to represent a solution as an exact superposition of elementary localized solutions running from the source or from the boundary.
I will consider two problems for the wave equation.

1) initial-value problem

\[ u_{tt} - c^2(u_{xx} + u_{yy} + u_{zz}) = \mathcal{F}(x, y, z, t), \]
\[ u|_{t=0} = 0, \quad u_t|_{t=0} = 0; \]

2) initial-boundary value problem in a half-space

\(-\infty < x, y < \infty, \quad z \geq 0,\)

\[ u_{tt} - c^2(u_{xx} + u_{yy} + u_{zz}) = 0, \]
\[ u(x, y, 0, t) = f(x, y, t) \]
\[ u(x, y, 0) = 0, \quad u_t(x, y, 0) = 0, \]

where \( f(x, y, t) = 0 \) when \( t \leq 0. \) The aim is to represent a solution as an exact superposition of elementary localized solutions running from the source or from the boundary.
Motivation

It can be effective if the source function $F$ or the time-dependent boundary data function $f$ are defined experimentally and they may have multiscaled structure; they have singularities and discontinuities or they may also be masked by a noise.
Motivation

It can be effective if the source function $F$ or the time-dependent boundary data function $f$ are defined experimentally and they may have multiscaled structure; they have singularities and discontinuities or they may also be masked by a noise. The most suitable mathematical tool for processing multiscaled functions is continuous wavelet analysis. We present here such a representation of the field which enables one to describe the propagation of the field in terms appropriate for processing of boundary data or source data. Constructed representation allows to extract from the propagating field only the part connected with propagation of singularities. We can also find the influence on the whole field of the field produced by some part of the boundary. The whole field may be reconstructed from ”parts”.
An example of data function for fixed time
Continuous wavelet analysis and solutions

A symmetric mother wavelet \( \psi(r) = \psi(|r|) \).

A function \( \psi(r) \in L_2(\mathbb{R}^2) \) is a mother wavelet in the case if
\[
C_\psi = \int_{\mathbb{R}^2} d^2k \frac{|\hat{\psi}(k)|^2}{|k|^2} < \infty,
\]
where \( \hat{\psi}(k) \) is the Fourier transform of \( \psi(r) \). A family of wavelets read:
\[
\psi_{ab}(r) = \frac{1}{a} \psi\left(\frac{r - b}{a}\right).
\]

Let \( \Psi(r, t) \) be a mother solution, \( \psi(r) \equiv \Psi(r, 0) \). A family of elementary solutions \( \Psi_{a\theta b}(r, t) \) is constructed from the mother solution as
\[
\Psi_{ab}(r, t) = \frac{1}{a} \Psi\left(\frac{r - b}{a}, \frac{t}{a}\right), \quad \psi_{ab}(r) \equiv \Psi_{ab}(r, 0).
\]
Wavelet transform and its properties

Wavelet transform $F(a, b)$ of a function $f(r)$, $f \in L_2(\mathbb{R}^2)$ reads

$$F(a, b) = \int_{\mathbb{R}^2} d^2 r \, f(r) \, \frac{1}{a} \psi\left( \frac{r - b}{a} \right).$$

If the wavelet $\psi(|r|)$ is localized near the point $r = 0$, then $\psi^{ab}(r)$ is localized near the point $b$ and the wavelet transform depends on the local properties of a function $f$ near the point $b$.

The wavelet transform in the Fourier domain reads:

$$F(a, b) = \frac{a}{(2\pi)^2} \int_{\mathbb{R}^2} d^2 k \, \hat{f}(k) \hat{\psi}(ak) \exp(ik \cdot b).$$

If the Fourier transform $\hat{\psi}$ is localized near the line $|l| = 1$ then $\hat{\psi}_{ab}$ is localized near the line $|k| = \frac{1}{a}$. The wavelet transform $F(a, b)$ contains an information of the Fourier content of $f(r)$ with wave numbers near $1/a$. 

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Reconstruction formula. Decomposition of solutions

The function \( f(r) \) can be reconstructed as follows

\[
f(r) = \frac{1}{C_\psi} \int_0^\infty \frac{da}{a^3} f_a(r), \quad f_a(r) = \int_{R^2} d^2b F(a, b) \psi\left(\frac{r - b}{a}\right)
\]

The reconstruction formula yields the decomposition of the solution of initial value problem in terms of elementary solutions

\[
u(r, t) = \frac{1}{C_\psi} \int_0^\infty \frac{da}{a^3} u_a(r, t), \quad u_a(r, t) = \int_{R^2} d^2b F(a, b) \Psi\left(\frac{r - b}{a}, \frac{t}{a}\right),
\]

\[
u(r, 0) = f(r).
\]
Decomposition of a solution in terms of cylindrical waves

\[ u_{tt} - c^2(u_{xx} + u_{yy}) = \delta(t)f(x, y), \]
\[ u|_{t<0} = 0, \ u_t|_{t<0} = 0, \]
\[ u(r, t) = \theta(t)v(r, t), \]
\[ v_{tt} - c^2(v_{xx} + v_{yy}) = 0, \]
\[ v|_{t=0} = 0, \ v_t|_{t=0} = f(r). \]

Let mother wavelet is symmetric.

\[ u(r, t) = \theta(t)v(r, t), \quad v(r, t) = \int_0^\infty \frac{da}{a^3} v_a(r, t), \]
\[ v_a(r, t) = \frac{1}{2C_\psi} \int_{\mathbb{R}^2} d^2b \ F(a, b)a \left( \Psi\left(\frac{r-b}{a}, \frac{t}{a}\right) - \Psi\left(\frac{r-b}{a}, -\frac{t}{a}\right) \right), \]
Decomposition of a solution in terms of cylindrical waves

Check initial conditions

\[ v_a(r, 0) = \frac{1}{2C_\psi} \int_{R^2} d^2b \ F(a, b) a \left( \dot{\Psi}\left(\frac{r-b}{a}, 0\right) - \Psi\left(\frac{r-b}{a}, 0\right) \right) = 0, \]

\[ \dot{v}_a(r, 0) = \frac{1}{2C_\psi} \int_{R^2} d^2b \ F(a, b) \frac{a}{a} \left( \dot{\Psi}\left(\frac{r-b}{a}, 0\right) + \dot{\Psi}\left(\frac{r-b}{a}, 0\right) \right), \]

According to the reconstruction formula:

\[ \dot{v}(r, 0) = \frac{1}{C_\psi} \int_0^\infty \frac{da}{a^3} \int_{R^2} d^2b \ F(a, b) \dot{\Psi}\left(\frac{r-b}{a}, 0\right) = f(r), \]

where \( F(a, b) \) is a wavelet transform of \( f(r) \) if a mother wavelet is taken as \( \dot{\Psi}(r, 0) \).
Examples of calculations of CWT. Singularities detection

A function \( f(r) \) and its wavelet transform \( F \) for fixed \( a \).
Examples of calculations of CWT. Propagation of waves

A function $f(r)$ and its wavelet transform $F$ for fixed $a$. 

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Examples of calculations of CWT. Propagation of waves

A function $f(r)$ and its wavelet transform $F$ for fixed $a$. 

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Examples of calculations of CWT. Propagation of waves

A function $f(r)$ and its wavelet transform $F$ for fixed $a$. 
Wavelet transform for small fixed $\alpha$
A function $\psi(r) \in L_2(\mathbb{R}^2)$ is a mother wavelet in the case if
$$C_\psi = \int_{\mathbb{R}^2} d^2 k \frac{\hat{\psi}(k)^2}{|k|^2} < \infty,$$
where $\hat{\psi}(k)$ is the Fourier transform of $\psi(r)$. A mother wavelet is directional if the essential numerical support of $\hat{\psi}(k)$ lies in an acute angle in the $k$ space with its vertex in the origin.

A family of wavelets is constructed from the mother wavelet $\psi(r)$ as follows
$$\psi_{a\theta b}(r) = \frac{1}{a} \psi \left( M^{-1} \frac{r - b}{a} \right), \quad M = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$
Elementary solutions. The reconstruction formula

Let $\Psi(r, t)$ be a mother solution, $\psi(r) \equiv \Psi(r, 0)$.
A family of elementary solutions $\Psi_{a\theta b}(r, t)$ is defined as follows

$$\Psi_{a\theta b}(r, t) = \frac{1}{a} \Psi \left( \frac{M^{-1} r - b}{a}, \frac{t}{a} \right), \quad \psi_{a\theta b}(r) \equiv \Psi_{a\theta b}(r, 0).$$

The wavelet transform $F(a, \theta, b)$ of a function $f(r)$, $f \in L_2(R^2)$:

$$F(a, \theta, b) = \int_{R^2} d^2 r \ f(r) \overline{\psi_{a\theta b}(r)}$$

The wavelet transform contains the information about the local properties of $f$ near $b$. It is determined by the Fourier transform $\hat{f}(k)$ of spatial frequencies $k$ located near the point with the polar coordinates $1/a, \theta$. 

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$$\Psi_{a\theta b}(r, t) = \frac{1}{a} \Psi \left( \frac{M^{-1}r - b}{a}, \frac{t}{a} \right), \quad \psi_{a\theta b}(r) \equiv \Psi_{a\theta b}(r, 0).$$

The wavelet transform $F(a, \theta, b)$ of a function $f(r), f \in L_2(R^2)$:

$$F(a, \theta, b) = \int_{R^2} d^2r \ f(r) \ \overline{\psi_{a\theta b}(r)}$$

The function $f(r)$ can be reconstructed by the formula

$$f(r) = \frac{1}{C_\psi} \int_0^\infty \frac{da}{a^3} \int_0^{2\pi} d\vartheta f_{a\theta}(r), \quad f_{a\theta}(r) = \int_{R^2} d^2b \ F(a, \theta, b) \psi_{a\theta b}(r).$$

$$u(r, t) = \frac{1}{C_\psi} \int_0^\infty \frac{da}{a^3} \int_0^{2\pi} d\vartheta u_{a\theta}(r, t), \quad u_{a\theta} = \int_{R^2} d^2b \ F(a, \theta, b) \Psi_{a\theta b}(r, t).$$
Decomposition of a solutions in terms of particle-like waves

\[ u_{tt} - c^2(u_{xx} + u_{yy}) = \delta(t)f(x, y), \quad u(r, t) = \theta(t)v(r, t), \]

\[ v_{tt} - c^2(v_{xx} + v_{yy}) = 0, \quad v(r, t) = \int_0^\infty \frac{da}{a^3} \int_0^{2\pi} d\theta v_a(r, t), \]

\[ v_a(r, t) = \frac{1}{2C_\psi} \int_{R^2} d^2b \, F(a, \theta, b) a \]

\[ \left( \Psi(M^{-1}r/b, t/a) - \Psi(M^{-1}r/b, -t/a) \right), \]

where \( \Psi(M^{-1}r/b, t/a) \) and \( \Psi(M^{-1}r/b, -t/a) \) represent wave packets moving in opposite directions, where \( F(a, \theta, b) \) is the wavelet transform of \( f(r) \) calculated with the mother wavelet \( \dot{\Psi}(r, 0) \).
Directivity diagrams. Examples of packet propagation

The scale $\alpha$ is fixed.
Directivity diagrams. Examples of packet propagation

The scale $a$ is fixed.
Directivity diagrams. Examples of packet propagation

The scale $a$ is fixed.