Physical wavelet defined on an observation plane and the Dreamlet
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Summary

Wavefield or seismic data are special data sets. They cannot fill the 4-D space-time in arbitrary ways. The time-space distributions must observe causality which is dictated by the wave equation. Wave solutions can only exist on the light cone in the 4D Fourier space. Physical wavelet is a localized wave solution by extending the light cone into complex causal tube. In this study we establish the link between the physical wavelet defined by Kaiser using AST (analytic signal transform) and the dreamlet (drumbeat-beamlet). We prove that dreamlet can be considered as a type of physical wavelet defined on an observation plane (earth surface or a plane at depth z during extrapolation). Causality (or dispersion relation) built into the wavelet (dreamlet) and propagator is a distinctive feature of physical wavelet which is advantageous for applications in wavefield decomposition, propagation and imaging. One example of dreamlet decomposition on seismic data is given.

Introduction

Physical wavelet was introduced by Kaiser (1993, 1994, 2003) as localized wave solutions to the wave equation by extending the solution to the complex space-time. In fact, the idea of localizing the wave solution by complex extension has been discussed by many authors (Felsen, 1976; Enziger and Raz, 1987; Heyman and Steinberg, 1987). Kaiser systematically developed the theory and termed it as “physical wavelet”. The wavelets derived in this way not only possess the properties of the wavelet, but also satisfy automatically the wave equation, which is a distinctive feature different from mathematical wavelets. This feature is very desirable when applied to physical problems such as wave propagation and imaging. From different approach but with similar physical insight, Wu et al. introduced a space-time wavelet formed by a tensor product of wavelet in time-domain (drumbeat) and beamlet in space domain and termed it as dreamlet (drumbeat × beamlet) (Wu et al., 2008, 2009; Wu and Wu, 2010). Dreamlet decomposition of seismic data is highly efficient and the propagator derived is very sparse. Although the dreamlet is defined on an observation plane, however, the dreamlet propagator derived based on the wave equation extends the wavelet into the full frequency-wavenumber space. In this study, we will establish the link between the physical wavelet and the dreamlet. We will prove that the dreamlet is a type of physical wavelet defined on an observation plane.

Brief overview on physical wavelet

Physical wavelet is defined as a localized elementary wave, which satisfies the wave equation and can be used as a decomposition atom in wavefield decomposition (wavelet transform) (See Kaiser, 1994). Here we concern only the acoustic (scalar) wave equation, and give a brief overview of the main concept using our notations. To construct such kind of wavelet, we start from the wave equation

$$(-c_o^2 + \nabla^2)u = 0$$

(1)

Transform the wavefield from the 4D space-time domain $x = (x,t) = (x_i, t_j)$ to the 4D Fourier domain

$$u(x) = \frac{1}{(2\pi)^4}\int_{\mathbb{R}^4} \int_{\mathbb{R}^4} e^{ip\cdot x - \omega \cdot t} \hat{u}(p)$$

(2)

where

$$p = (p, p_0) = (p, k) \in \mathbb{R}^4$$

(3)

is the 4-D wavenumber, and the dot product (Lorentz-invariant inner product) is defined as

$$p \cdot x = p \cdot x - p_0 x_0 = p \cdot x - \omega t$$

(4)

with $p_0 = k = \omega / c$ as the normalized frequency (absolute wavenumber), and $x_0 = ct$ as the normalized time. We define the 3D wavenumber vector as

$$p = (\xi, \zeta) \in \mathbb{R}^2$$

(5)

with $\xi$ and $\zeta$ as the horizontal and vertical wavenumbers, respectively. From the wave equation, the 4D wavenumber $p$ must satisfy the dispersion relation:

$$p^2 = p \cdot p = |p|^2 - p_0^2 = |p|^2 \left(\frac{\omega^2}{c^2}\right) = 0$$

(6)

Thus the solution can only exist on the light cone:

$$C_{\pm} = \{ p : p^2 = 0, p \neq 0 \} = C_{\pm} = C_{\pm}$$

(7)

Therefore, the solution has the form

$$\hat{u}(p) = 2\pi \delta(p^2) u(p)$$

(8)

where

$$u_z(p) = u(p, \pm k), \quad k = |p|$$

(9)

$$\delta(p^2) = \delta(p_0^2 - k^2 - p_0^2 + k) = \delta(p_0 - k) \delta(p_0 + k)$$

(10)
Finally we obtain the solution

\[
  u(x) = \int_{\mathbb{C}} \frac{d^4 p}{16\pi^2 k} \left[ e^{i(p \cdot x - \omega t)} u(p, \omega) + e^{i(p \cdot x + \omega t)} u(p, -\omega) \right] 
\]

\[
  = \int_{\mathbb{C}} d^4 p \exp^{i(p \cdot x)} u(p) 
\]

where

\[
  d^4 p = \frac{d^4 p}{16\pi^2 k} \tag{12} 
\]

is the Lorentz-invariant measure on \( \mathbb{C} \). We see that the wave solution (11) is in a form of plane wave superposition in the 4D Fourier domain. In order to express the wave solution as a superposition of localized waves, Kaiser (e.g. 1994) applied an AST (analytic signal transform) to \( u(x) \), resulting in an extension of \( u(x) \) from real space-time (the light cone) to a causal tube \( T = \{(x + iy) \in \mathbb{C}^4 : y^2 > 0\} \) in complex space-time

\[
  \tilde{u}(x + iy) = \frac{1}{\pi i} \int_{-\infty}^{\infty} d\tau e^{i(x + iy)\tau} (x - i\tau) \]

\[
  = \int_{\mathbb{C}} d^4 p \Theta(p \cdot y) e^{i(p \cdot x + iy)} u(p) \tag{13} 
\]

\[
  = \int_{\mathbb{C}} d^4 p \exp^{i(p \cdot x)} \psi_j(p) u(p) 
\]

where \( \Theta \) is the unit step function, and

\[
  \psi_j(p) = 2k e^{-\zeta} \Theta(p \cdot y) \exp\left[i(p \cdot x - iy)\right] \tag{14} 
\]

is an acoustic wavelet of order \( \alpha \) in the Fourier domain. From (13) we see that the AST can be looked as a windowing in the Fourier domain (windowed Fourier transform). Figure 2 shows some examples of the time-domain waveform (for a fixed space location) of different orders \( \alpha = 3, 10, 15, 50 \). In the same way, we can also show the space localization at different times.

As discussed by Kaiser (1994, 2006), the inner product of a wavefield with a physical wavelet in (13) is windowing process in the 4D Fourier domain with a window function \( h(p \cdot y) \). The window function is defined in the causal tube which is the analytic extension of the light cone. Points in the causal tube have projections on the light cone in the form of windowing. Therefore, the net effect of AST is windowing on the light cone (Figure 3). Since the light cone is the causality surface (or dispersion surface), a hyper-surface in 4D Fourier space, so the physical wavelets automatically satisfy the wave equation. From

\[
  p \cdot y = \frac{\xi}{\gamma} \gamma y_{r} + \zeta y_{j} - k_{j} y_{0} \tag{15} 
\]

where subscript “T” stands for “transversal”, and the exponential term in (13), we see the pulse width (waveform) is controlled by \( y_{0} \) and the beamwidth and steering are parameterized by \( (y_{r}, y_{j}) \).

![Figure 2](image2.png)

Figure 2 Physical wavelet along the time-axis (at \( t=0 \)) for \( \alpha = 3, 10, 15, 50 \). Solid lines are the real part, and dotted lines are the imaginary part.

![Figure 3](image3.png)

Figure 3 Windowing on the light cone, representing directional wavepacket (pulsed-beam)

**Physical wavelet and the dreamlet**

**Dreamlet**: A type of physical wavelet defined on the observation plan.
Physical wavelet and the dreamlet

"Dreamlet" is defined as a tensor product of a time-frequency atom "drumbeat", and the space-wavenumber atom "beamlet" in decomposing the wavefield on an observation plane. Drumbeat × beamlet = dreamlet. The observation plane could be the earth surface or a subsurface at depth z during wavefield extrapolation. The time-frequency localization of the dreamlet is treated separately from the space-wavenumber localization. A dreamlet atom is in the form

\[ d_{\tau\omega}(x,t) = g_{\tau\omega}(t)b_{\tau\omega}(x) \]  
where

\[ g_{\tau\omega}(t) = W(t - \tau)e^{-i\omega t} \]  
is a \( t-f \) atom (drumbeat) with \( W(t) \) as a smooth window function, and

\[ b_{\tau\omega}(x) = B(x - \tau)e^{i\tau x} \]  
is an \( x-\xi \) atom (beamlet) with \( B(x) \) as a bell function. In this section, \( x \) represents a horizontal position on the observation plane, not to be confused with the 4D space-time location in the previous section. The bars over letters signify the variables are window centers, so are local variables. Note that the phase terms in the \( t-f \) atom and the \( x-\xi \) atom have opposite signs. This is consistent with the Lorentz-invariant inner product when performing the decomposition. The corresponding frequency-wavenumber domain expressions are

\[ g_{\tau\omega}(\omega) = W(\omega - \overline{\omega})e^{i\omega \tau} + W(\omega + \overline{\omega})e^{-i\omega \tau} \]

\[ b_{\tau\omega}(\xi) = B(\xi - \overline{\xi})e^{i\xi \tau} \]  

On the observation plane, the integration on the light cone (in the 4D Fourier space) will be in a different form

\[ u(x) = \int_{c} dp e^{i p x} u(p) \]  
with a new measure

\[ dp = \frac{d^{2} \xi dk}{16\pi^{3} |k^{2} - |\xi|^{2}|^{3/2}} \]

The wavelet in this case will be in a form of

\[ \psi(p) = \Theta(k^{2} - \xi^{2})d_{\tau\omega}(\xi,\omega) \]

When we do the local decomposition, the local homogeneous medium approximation is assumed so the concept of the light cone can be applied locally. Although the beamlet transform is applied on the observation plane, however, the beamlets are defined in the full wavenumber space through the dispersion relation. As shown in Figure 4, the window \( B(x - \tau) \) on the observation plane (here the \( x \)-axis for the 2D case) is a cross-section of the whole space window marked as the green disk (assuming isotropic windows). Through the dispersion relation, we can relate the local horizontal wavenumber \( \xi \) to the local propagation direction \( \mathbf{p} = (\xi, \zeta) \) for a given \( \omega \)

\[ \xi = \pm \sqrt{(\omega/c)^{2} - \zeta^{2}} \]

where \( \Theta \) is the propagation angle with respect to the \( z \)-axis. For one-way propagation, we take only the positive sign. Therefore, the beamlet defined in (18) by the local Fourier transform along the horizontal plane represents a local beam

\[ b_{\tau\omega}(x,z) = B^{2}(\tau, z_{0})e^{i(T^{+} - \tau^{+} - z_{0} \zeta)} \]  

where \( B^{2}(\tau, z_{0}) \) is window in green color centered at the grid point on the observation plane (Figure 4). The beamlet propagator in (24) is implemented by a sparse matrix operation (see Wu et al, 2008a).

Figure 4 The window defined on the observation plane (red segment) (horizontal) and window for the whole space (green disk).

Now we discuss the parameterization in the dreamlet construction in comparison with the physical wavelet. The light cone in the 4D space-time or the 4D Fourier space can be specified by triplet vectors. As shown in (11), the frequency can be parameterized by \( p = (\xi, \zeta) \in R^{1} \). In that case, the frequency is determined by \( k = |p| \). In the case of dreamlet decomposition, the light-cone integration is parameterized by \( \xi \) and \( k \). In the latter case, the vertical wavenumber \( \zeta \) is determined by the dispersion relation. In the physical wavelet, the pulse width, beam width and steering is controlled by the parameter of AST, i.e. \( y = (r, s) \), where \( s \) controls the pulse (wavepacket) width, and \( r \) affects the beam parameters. In the dreamlet decomposition, the window in time controls the duration of drumbeat (pulse width), and the window on the observation plane determines the beam width. The beam direction is specified by \( \xi / k \).

Examples of dreamlet decomposition on seismic data
Physical wavelet and the dreamlet

Here we give one example of the data decomposition using dreamlets to see the flexibility of dreamlet in selecting the time window widths to fit the wavepacket width in the data. Figure 5 shows the poststack data of SEG 2D salt model. Figure 6 gives the dreamlet decomposition of the SEG salt data by local exponential frames. The top and bottom panels are for left- and right-propagating coefficients respectively. In this figure, the horizontal and vertical coordinates are for $(\tau, \xi)$ and $(\tau, \omega)$ respectively. In Figure 7 we show dreamlet decompositions using different window widths for the drumbeat: top: 32 points; mid: 16 points, bottom: 8 points. In this figure, along the horizontal: the first level of grid is the local wavenumber $\xi$, and the fine grid in the second level is for $\tau$; Along the vertical: first level is for $\omega$, and fine grid is for $\tau$. We see that coefficient distributions on different local frequency $\omega$ are changing with different drumbeat widths. This gives the flexibility of adapting the decomposition according to the wavepacket width in the data. Because the dreamlet decomposition of seismic data takes the advantage that causality is the inherent property of the data, the decomposition is very efficient (Wu et al., 2008).

Conclusions

Wave solutions can only exist on the light cone in the 4D Fourier space. Physical wavelet is a localized wave solution by extending the light cone into complex causal tube. Dreamlet can be considered as a type of physical wavelet defined on an observation plane (earth surface or a plane at depth $z$ during extrapolation). Causality (or dispersion relation) built into the wavelet (dreamlet) and propagator is a distinctive feature of physical wavelet which is advantageous for applications in wave data decomposition, propagation and imaging.

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