The far-field approximation in seismic interferometry

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Summary

Green’s function retrieval through seismic interferometry can be derived based on the Rayleigh’s reciprocity theorem of the correlation type. However, the common underlying assumption of theory that the sources are in the far field limits the technology to extracting correctly only the high frequency part of the Green’s function in an open system. This critical approximation can be eliminated using the exact boundary integral equation method. Without the far-field approximation, a crosscorrelation kernel is proposed to recover the exact Green’s function. Symmetric difference kernels are analytically constructed for sources on a plane or on a circle and can be reduced to the known Dirac delta kernel under the far-field approximation.

Spurious arrivals can be artificially generated in the construction of Green’s function using wavefield crosscorrelation if a small scattering object is present in the medium. For far-field sources, if the point scattering model is correct, the spurious arrival can be eliminated; however, if the sources are not in the far field, the crosscorrelation kernel must also be used.

Introduction

Seismic interferometry (e.g., Schubert, 2001; Snieder, 2004; Wapenaar, 2004; Wapenaar and Fokkema, 2006) has been a thriving field and a rich body of novel and interesting applications related to the Green’s function retrieval have been produced and we refer to excellent review articles and books (Larose et al., 2006; Wapenaar, Draganov and Robertsson, 2008; Shuster, 2009; Snieder et al., 2009a). However, the theory, in particular for the open system, invoked an assumption that the illuminating sources are in the far field that may be violated. This far-field approximation is convenient for both theory and application. The consequence of this critical approximation is that the retrieved Green’s function contains incorrect amplitude (Zheng, 2010) and it may produce spurious arrivals even the correct scattering model is used. In this article we will see the far-field approximation can be removed through exact boundary integral equation method and investigate the effect of this approximation numerically.

Cross-correlation kernels for exact Green’s function retrieval

The Green’s function retrieval through seismic interferometry can be understood by a surface integral (Figure 1) in the frequency \( \omega \) domain using the Rayleigh’s reciprocity of the correlational type (e.g., Wapenaar and Fokkema, 2006; Shuster, 2009):

\[
2i \ln G(\mathbf{x}_s | \mathbf{x}_s, \omega) = \oint_{\partial \mathbb{D}} \left[ G(\mathbf{x}' | \mathbf{x}_s, \omega) \frac{\partial \overline{G}(\mathbf{x}' | \mathbf{x}_s, \omega)}{\partial n'} \right] d^2 \mathbf{x}'
\]

in which \( G(\mathbf{x}' | \mathbf{x}_s, \omega) \) and \( \overline{G}(\mathbf{x}' | \mathbf{x}_s, \omega) \) are the Green’s functions for the heterogeneous medium in \( \mathbb{D} \) which satisfy differential equations

\[
V^\ast G(\mathbf{x}' | \mathbf{x}_{s,0}, \omega) + k^2 G(\mathbf{x}' | \mathbf{x}_{s,0}, \omega) = -\delta(\mathbf{x}' - \mathbf{x}_{s,0}), \quad \mathbf{x}' \in \partial \mathbb{D}
\]

and the over bar denotes the complex conjugate and \( \partial / \partial n' \) is the directional derivative along the outward normal of the boundary \( \partial \mathbb{D} \) with respect to position \( \mathbf{x}' \in \partial \mathbb{D} \). For simplicity, we neglect the source time function in the formulation. The right hand side of (1) can be interpreted as focusing/backpropagating the wavefield on \( \partial \mathbb{D} \) generated by a source at \( \mathbf{x}_s \) to the observation point \( \mathbf{x}_b \). Based on a physical argument using the time reversal acoustics (Derode et al., 2003), the left hand side of (1) corresponds to \( G(\mathbf{x}_b | \mathbf{x}_s, t) - G(\mathbf{x}_s | \mathbf{x}_b, -t) \) in the time \( t \) domain, which consists of the causal and the anticausal Green’s functions. Let \( p_s(\mathbf{x}') = G(\mathbf{x}' | \mathbf{x}_s, \omega) \) and \( \overline{p}_s(\mathbf{x}') = \overline{G}(\mathbf{x}' | \mathbf{x}_s, \omega) \) be the wavefield on \( \partial \mathbb{D} \) produced by a source at \( \mathbf{x}_s \). Likewise, \( p_b(\mathbf{x}') = G(\mathbf{x}' | \mathbf{x}_b, \omega) \) and \( \overline{p}_b(\mathbf{x}') = \overline{G}(\mathbf{x}' | \mathbf{x}_b, \omega) \) are due to the source at \( \mathbf{x}_b \). Equation (1) can be written in a compact form

\[
2i \ln G(\mathbf{x}_s | \mathbf{x}_s, \omega) = \langle p_s, \overline{p}_s \rangle - \langle q_s, p_b \rangle
\]

in which \( \langle f, g \rangle = \int_{\partial \mathbb{D}} f(\mathbf{x}) \overline{g}(\mathbf{x}) d^2 \mathbf{x} \) is the inner product defined on the boundary. In previous studies (e.g., Wapenaar, Fokkema and Snieder, 2005; Wapenaar and Fokkema, 2006; Snieder, Wapenaar and Wegler, 2007), the integral can be simplified in an open system using the far-field approximation for a spherical \( \partial \mathbb{D} \) with a large radius such that \( q_{s,b}(\mathbf{x}') = i \omega c^{-1} \cdot j p_{s,b}(\mathbf{x}') \), \( \mathbf{x}' \in \partial \mathbb{D} \). If the wave propagation speed \( c(\mathbf{x}') \) is constant, we obtain

\[
2i \ln G(\mathbf{x}_s | \mathbf{x}_s, \omega) = -2i k \langle p_s, p_b \rangle, \quad k = \omega / c
\]

which is referred as Green’s function retrieval from far-field correlations. By the source-receiver reciprocity in the Green’s function, \( p_s(\mathbf{x}') \) and \( p_b(\mathbf{x}') \) can be interpreted as
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the wavefields recorded at \( \mathbf{x}_A \) and \( \mathbf{x}_B \), respectively, produced by a source at \( \mathbf{x}' \in \partial \Omega \). Each source contributes equally to the Green’s function retrieval on the left hand side. Next, we show that one can still relate the integral (1) to the crosscorrelation of the wavefield as in (3) even when the boundary is not in the far-field regime.

\[
\begin{align*}
\text{Figure 1} & \quad \text{Heterogeneous medium bounded by a surface } \partial \Omega. \\
\end{align*}
\]

It is well known in the studies of the Huygens principle using single-layer and double-layer potential representations that the surface values \( p_a \) and \( q_a \) (or \( p_b \) and \( q_b \)) are generally not independent (e.g., Baker and Copson, 1950; Kupradze, 1963; Colton and Kress, 1983). By exploiting this property, Zheng (2010) gave a rigorous mathematical derivation without the far-field approximation for the Green’s function retrieval for sources at all distances by the boundary integral equation method in the open system where the Sommerfeld radiation condition is ensured at the infinity. In his derivation only wavefield correlation (no wavefield gradient) is involved in the crosscorrelation but there is a correlation kernel.

\[
2i \text{Im} \mathcal{G}(\mathbf{x}_A; \mathbf{x}_B; i\omega) = \langle p_A, C p_b \rangle - \langle C p_A, p_b \rangle 
\]

\[
= \langle p_A, (C - C^*) p_b \rangle = \langle p_A, W p_b \rangle 
\]

(4)

in which \( p_{a,b} = C p_{a,b} \) and \( C^* \) is the adjoint of \( C \). If the boundary \( \partial \Omega(D) \) is a circle of radius \( R \) in 2D (a cylinder), one can show that the kernel \( W \) is a symmetrical difference kernel whose action on a function is a convolution.

\[
(W p_b)(\phi) = \frac{1}{2\pi} \int_0^{2\pi} W(\phi - \phi') p_b(\phi') d\phi' 
\]

and \( W(\phi) \) can be constructed by its Fourier series coefficient

\[
W_n = \frac{4i}{\pi R} \left[ H_n(kR) \right]^2, \quad n = 0, \pm 1, \pm 2, \ldots 
\]

(5)

of the basis function \( e^{in\phi} \), where \( H_n \) is the n-th order Hankel function of the first kind and \( k \) is the wavenumber in the exterior medium outside of \( \partial \Omega \). For the far-field case, \( kR \gg 1 \), our crosscorrelation kernel reduces to the known result, \( W(\theta) = 2ik\delta(\theta) \), where \( \delta(\theta) \) is the Dirac delta function. It can be shown in general the kernel only depends on the geometry of the boundary and the medium property of the exterior medium. If the boundary is in arbitrary geometry, the kernel becomes a matrix which can be solved by the boundary element method and as such operator \( C \) will be a square matrix and \( C^* \) is the Hermitian transpose of \( C \). We take the interpretation that \( p_A(\mathbf{x}') \) and \( p_B(\mathbf{x}') \) are wavefields at \( \mathbf{x}_A \) and \( \mathbf{x}_B \), respectively, due to a boundary source at \( \mathbf{x}' \). Clearly, if \( W = \{w_i\} \) is a diagonal matrix, then the backpropagation is a weighted correlation, \( \sum_i w_i p_i^j \bar{q}_i^j \) in which \( i \) is the index of the source. However, if \( W \) is a full matrix, the backpropagation integral becomes \( \sum_i \sum_j w_i^j p_i^j \bar{q}_i^j \) and \( i \) and \( j \) are the source indices. In this case, the wavefield \( p_i^j \) at \( \mathbf{x}_A \) due to the \( i \)th source on the boundary is crosscorrelated with the wavefield \( p_i^j \) at \( \mathbf{x}_B \) due to the \( j \)th source, which is in sharp contrast with the previous thought that only wavefields at two locations from the same source (\( i = j \)) are crosscorrelated. If there are \( N \) sources on the boundary in total, we need to do cross-correlation

\[
\left( \begin{array}{c}
\mathbf{W} \\
\end{array} \right) = \left( \begin{array}{c}
N \\
\end{array} \right) 
\]

(6)

whose action on a function represents convolution.

Numerical Examples

Example 1: If the surface is a cylinder with radius \( R \) and the exterior medium is unbounded and homogeneous and has a wavenumber \( k \) (Figure 2a). Figure 2a shows the geometry of two sources \( A, B \) and receivers in our first numerical example. All the receivers are placed on the circle surrounding these two sources. One scattering point is embedded in the homogeneous background medium. The wavefields at receivers are generated using first-order born approximation. The velocity in the background media is \( 2000 \) m/s. We use a Ricker wavelet with center frequency 50 Hz. There are totally 629 receivers evenly distributed on the circle with radius \( R=800 \) m. Source \( A \) is located at \( (-500, -100) \) m and source \( B \) \( (+500, -100) \) m. The scattering point \( C \) is put at \( (0, +600) \) m. The diffraction index equals to 50. Figure 2b shows the kernel for center frequency at one receiver \( (x_g, z_g) = (-800, 0.0) \) m. Shown in Figure 3 is the comparison between the retrieved Green’s function using cross-correlation kernels in equation 4 and that without using kernels (equation 3). The analytical Green’s function using first-order born approximation consists of the direct wave from \( A \) to \( B \) and the scattered wave from the scattering point \( C \). Here we only show the scattered wave from the scattering point. The analytical Green’s function
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(blue) is used as benchmark. Not surprisingly, the retrieved Green’s function without using kernels yields incorrect amplitude of the scattered wave albeit the small difference. The retrieved Green’s function using cross-correlation kernels in equation 4 exactly overlaps the analytical Green’s function.

![Figure 2](image1.png)

**Figure 2.** Cylinder surface (a) and corresponding cross-correlation kernel (b) for receiver at (-800, 0) m.

receivers are located from -4000 m to +4000 m with a spatial interval 8 m. Source A is located at (0.0, -400) m, and source B (0.0, -1200) m. The scattering point C is put at (-400, -800) m. The diffraction index equals to 100. Figure 4b shows one cross-correlation kernel for center frequency at receiver (x_g, z_g)=(0.0, 0.0). Shown in Figure 5 is the comparison between the retrieved Green’s function using cross-correlation kernels and that without kernels. The analytical Green’s function is used as benchmark. We can see that the retrieved Green’s function without using kernels again yields incorrect amplitude of the scattered wave. With the cross-correlation kernel (6) we can exactly recover the Green’s function.

![Figure 4](image2.png)

**Figure 4.** Infinite plane surface (a) and corresponding cross-correlation kernel for receiver at (0, 0) m.

**Example 3: spurious arrival without the kernel**

It is observed that an unphysical or a spurious arrival can be generated when a point scatter is present in the medium if the Born scattering model is used. Previous studies (Snieder et al., 2008; Snieder, Sánchez-Sesma and Wapenaar, 2009b; Wapenaar, 2009) resolved this “paradox” by showing that the linear Born scattering is not adequate and the correct or nonlinear scattering model must be used to eliminate the spurious arrival under the far field approximation and moreover they have demonstrated that the Green’s function retrieval is related to the optical theorem (e.g., Newton, 1976). This is consistent with the analytical study by (Sánchez-Sesma and Campillo, 2006; Sánchez-Sesma et al., 2006) in which a cylindrical scatterer is illuminated by incident plane waves and they showed that no spurious arrival is produced. Here we want to show that if the sources are not in the far field, even the correct scattering model is used but not with the crosscorrelation kernel, the spurious arrival still exists. The numerical model setup (Fig.6) has a cylindrical inclusion at the origin.
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with density $\rho = 2 \times 10^4$ kg/m$^3$ (not realistic) and velocity $c_1 = 6500$ m/s. The density and velocity of the background are $\rho_0 = 1000$ kg/m$^3$ and $c_0 = 1500$ m/s, respectively. The radius of the sources is $R = 1000$ m. Two receivers are at $(-400, 300)$ and $(400, 300)$ in Cartesian coordinates. For each source, the source time function is a Ricker wavelet with central frequency $f_0 = 10$ Hz.

crosscorrelation kernel. Scattering by a small scatterer can produce spurious arrivals if the scattering theory is not exact. If the scattering theory is exact and the sources are in the far field, there will be no such arrivals. However, if the medium is illuminated by sources that are not in the far field, both the correct scattering model and the crosscorrelation kernel must be used to eliminate the unphysical arrival.

Figure 5. Comparison between theoretical Green’s function for scattered waves using first-order Born approximation (blue) and retrieved Green’s functions (red dotted dash) from seismic interferometry without kernels (a) and that using cross-correlation kernels (b).

Conclusions
In this paper, we have pointed out that the previous theory with the far-field approximation on the Green’s function retrieval yielded incorrect amplitude in an open system through numerical examples. This widely used approximation can be removed by the integral equation method. If deterministic sources were used, we have shown that in order to exactly retrieve the Green’s function we need correlate wavefields not only between receivers for the same source but also between different sources. Finally, we remark that for a great number of applications, the Green’s function retrieval technique is being applied to diffuse wavefields generated by noise sources and as such the crosscorrelation kernel should still be used if the source locations can be estimated. For a boundary with general geometric shape on which the sources are excited, the boundary element method can be used to solve for the

Figure 6. Scattering geometry.

Figure 7: Comparison between the theoretical Green’s function (thick gray line) and the retrieved Green’s function (thin black line) by wavefield crosscorrelation with (A) and without (B) the kernel. Notice the spurious arrival around time zero in (B), albeit its small amplitude.

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