Lateral velocity variation related correction in asymptotic true-amplitude one-way propagators

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Summary

Much work has been done on the vertical velocity variation related amplitude correction term in the asymptotic true-amplitude one-way wave equation, but the lateral velocity variation related correction has not received much attention, even being excluded in some asymptotic true-amplitude one-way propagator formulations. We investigate the effects of different amplitude correction terms in the asymptotic true-amplitude one-way wave equation, especially the effect related to the lateral velocity variation. We derive a dual-domain wide-angle screen type asymptotic true-amplitude one-way propagator and evaluate two implementations of the amplitude correction. Numerical examples show that the lateral velocity variation related correction term can play a significant role in the asymptotic true-amplitude one-way propagator. Optimization of the expansion coefficients in the asymptotic true-amplitude one-way propagator can improve both the amplitude and phase accuracy for wide-angle waves.

Introduction

Conventional one-way wave equations do not pay much attention to the correctness of amplitude information and cannot provide accurate amplitudes even at the level of leading order asymptotic WKBJ or ray theory (Zhang et al., 2003). WKBJ amplitudes have long been introduced into the conventional one-way wave propagator (e.g. Clayton and Stolt, 1981). Traditionally the WKBJ solution was derived by asymptotic approximation in smoothly varying \(v(z)\) media (e.g. Morse and Feshbach, 1953; Aki and Richards, 1980). It has also been obtained by approximately factorizing the full-wave operator into one-way wave operators in heterogeneous media (Zhang, 1993). It was theoretically proved that the obtained one-way wave equation could provide the same amplitude as the full wave equation in heterogeneous media in the sense of high-frequency asymptotics (e.g., Zhang, 1993). In this sense, they are called the "true-amplitude" one-way wave equation (Zhang et al., 2003). To precisely represent the physics, we refer to it as the asymptotic true-amplitude one-way wave equation. The WKBJ solution was also derived from the conservation of energy flux in smoothly varying \(v(z)\) media and extended to general media using local wavenumber/angle domain propagators by introducing the concepts of a “transparent boundary condition” and a “transparent propagator” (Wu and Cao, 2005; Cao and Wu, 2005, 2006, 2008; Luo et al., 2005).

With the asymptotic true-amplitude one-way propagator, improved image amplitude is obtained from single-shot migration in some smoothly varying models (Zhang et al., 2003; Zhang et al., 2005). As for the wavefield amplitude, much attention is paid to the vertical velocity variation related amplitude correction term in the literature, while the lateral velocity variation related correction term has rarely been studied, even having been excluded in some asymptotic true-amplitude one-way propagator formulations. In this paper, we analyze the effects of amplitude correction terms in the asymptotic true-amplitude one-way propagator, especially the effect related to the lateral velocity variation, by comparing the wavefield amplitude from the one-way propagator with that from full-wave modeling.

We illustrate these with scalar wave propagation. A finite-difference scheme is used to solve the full wave equation. The implementation of the asymptotic true-amplitude one-way propagator in Zhang et al. (2005) uses a space-domain finite-difference scheme. Considering the advantage of dual-domain extrapolators compared with pure finite-difference extrapolators in the conventional one-way propagation, we prefer to use the dual-domain extrapolator in the true-amplitude one-way propagation.

The outline of the paper is the following. First we derive a dual-domain wide-angle screen type asymptotic true-amplitude one-way propagator including the lateral velocity variation correction term and compare two implementations of the amplitude correction. Then we show some numerical examples to analyze the effect of amplitude correction terms in the asymptotic true-amplitude one-way propagator.

Implementations of the asymptotic true-amplitude one-way propagator

Asymptotic true-amplitude one-way wave equation

The conventional one-way wave equation for generally heterogeneous \(v(x, y, z)\) media can be written as

\[
\begin{align*}
\left(\frac{\partial}{\partial z} - i\Lambda_0\right)P(x, y, z, \omega) &= 0, \\
P(x, y, z, 0, \omega) &= \frac{1}{2\pi} \Lambda_0^{-1} \delta(x - x_s)
\end{align*}
\] (1)

where \(x_s=(x, y)\) denotes transverse/lateral coordinates, \(P(x, y, z, \omega)\) is pressure, and the square-root operator \(\Lambda_0\) is

\[
\Lambda_0(x, y, z) = \frac{\omega}{v(x, y, z)} \sqrt{1 + \frac{\nabla^2(x, y, z)}{\omega^2 - \frac{\partial^2}{\partial x^2}}}.
\] (2)

with

\[
\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.
\] (3)

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Asymptotic true-amplitude one-way propagators

The asymptotic true-amplitude one-way wave equation for generally heterogeneous \((x, y, z)\) media proposed by Zhang (1993) can be explicitly written in the frequency domain as (see also Zhang et al., 2005)

\[
\frac{\partial}{\partial \omega} - i \Lambda \Gamma P(x, y, z, \omega) = 0 \quad , \quad \text{(4)}
\]

\[
P(x, y, z = 0, \omega) = \frac{1}{2i} \Lambda^{-1} \delta(x - x_0)
\]

where \(\Lambda\) is a square-root operator

\[
\Lambda(x, z) = \frac{\omega}{\sqrt{1 + \frac{\omega^2}{\partial^2 x_s} + \frac{\omega}{\partial^2 x_s} \frac{\partial}{\partial x_s}}}
\]

and

\[
\Gamma(\mathbf{x}_r, z) = \frac{1}{2} \frac{\partial^2}{\partial \omega^2} \left[ x_s \right]^2 \left[ x_s \right] = \left\{ \frac{\partial^2}{\partial x_s^2} + \frac{\partial}{\partial x_s} \frac{\partial}{\partial x_s} \right\}
\]

\[
\left( \omega \delta(x - x_0) \right)
\]

Compared with the conventional one-way wave equation (1), the asymptotic true-amplitude one-way wave equation (4) includes an extra amplitude correction term \(\Gamma\); besides that, the square-root operator \(\Lambda\) includes the contribution from lateral velocity variation which is neglected in the conventional one-way propagator \(\Lambda_\circ\). Both terms can influence the amplitude of the one-way propagator.

**Correction at every extrapolation step**

The straightforward way to implement the above asymptotic true-amplitude one-way propagator (4) is, at each extrapolation step, to solve first the equation without the correction term \(\Gamma\)

\[
\frac{\partial}{\partial \omega} P(x, y, z, \omega) = \Gamma P(x, y, z, \omega)
\]

\[
P(x, y, z = 0, \omega) = \frac{1}{2i} \Lambda^{-1} \delta(x - x_0)
\]

then apply the \(\Gamma\) term correction,

\[
\frac{\partial}{\partial \omega} P(x, y, z, \omega) = \Gamma P(x, y, z, \omega)
\]

Equation (7) has the same form as the conventional one-way propagator (1). Similar to the case for the square root operator \(\Lambda_\circ\), a first-order Padé approximation (e.g., Collins, 1989) can be used to expand the square root operator \(\Lambda\) (5),

\[
\Lambda'(x, z) = k(x, z)
\]

\[
\left[ \frac{a_0 \left( \frac{1}{k'(x, z)} \frac{\partial^2}{\partial x_s^2} + \frac{\partial}{\partial x_s} \frac{\partial}{\partial x_s} \right)}{1 + b_1 \left( \frac{1}{k'(x, z)} \frac{\partial^2}{\partial x_s^2} + \frac{\partial}{\partial x_s} \frac{\partial}{\partial x_s} \right)} \right]
\]

where

\[
k(x, z) = a_0 \nu(x, y, z)
\]

As in the conventional one-way propagator case, we can obtain the coefficients \(a_0 = 0.5\) and \(b_1 = 0.25\). Considering the advantage of dual-domain extrapolators compared with pure finite-difference extrapolators for conventional one-way propagation, we prefer to use dual-domain propagators for the true-amplitude one-way propagation. By a derivation similar to that for the conventional one-way propagator (Xie and Wu, 1998), we can obtain a dual-domain wide-angle screen type true-amplitude one-way propagator

\[
\Lambda'(x, z, \omega) = \sqrt{\frac{\omega^2}{\nu_0^2} + \frac{\partial^2}{\partial x_s^2} + \frac{\partial}{\partial x_s} \left( \frac{1}{\nu_0^2} \right)}
\]

\[
\left[ \frac{a_0 \left( \frac{1}{k'(x, z)} \frac{\partial^2}{\partial x_s^2} + \frac{\partial}{\partial x_s} \frac{\partial}{\partial x_s} \right)}{1 + b_1 \left( \frac{1}{k'(x, z)} \frac{\partial^2}{\partial x_s^2} + \frac{\partial}{\partial x_s} \frac{\partial}{\partial x_s} \right)} \right]
\]

in which \(\nu_0 = \nu_0(z)\) is the reference velocity at depth level \(z\) and \(m = \nu(x, y, z)\) is the lateral velocity contrast. This propagator has the same form as the conventional wide-angle screen propagator; therefore it can be implemented in a similar way. The difference is that the wide-angle finite-difference correction term in the asymptotic true-amplitude wide-angle screen type propagator (11) includes the contribution from the lateral velocity gradient. These extra terms require only a little extra computation. At each step, after propagation, the \(\Gamma\) correction (8) can be implemented by a space-domain finite-difference scheme. The \(\Gamma\) term correction is the main extra computation for the asymptotic true-amplitude one-way propagator compared with the conventional one-way propagator.

**Flux transparent propagator**

The above implementation for the asymptotic true-amplitude one-way propagator needs to apply the \(\Gamma\) term correction at every extrapolation step down to the target area, which will be time-consuming if we are interested in the amplitude behavior of the pressure only for very deep targets, e.g. in sub-salt regions. By a variable change in the asymptotic true-amplitude one-way propagator (4), another form of the true-amplitude one-way propagator can be obtained (e.g., Zhang et al., 2005),

\[
\left( \frac{\partial}{\partial \omega} - i \Lambda \Gamma P(x, y, z, \omega) = 0 \quad , \quad \text{(12)}
\]

\[
P(x, y, z = 0, \omega) = -\frac{1}{2i} \Lambda^{-1} \delta(x - x_0)
\]

where

\[
F = \Lambda^{-1} P.
\]

The amplitude correction term \(\Lambda\) does not explicitly show up in this new true-amplitude one-way equation. It has the same form as the conventional one-way wave equation (1) but with a different propagator which includes the contribution from lateral velocity variation and also a different boundary condition which corresponds to using an energy-flux normalized source instead of a pressure source to excite the wave. The pressure field \(P\) can be obtained using its relation with the flux (13). With this new scheme, the target-oriented pressure field can be obtained with relatively low cost. The downward continuation of \(F\) makes the energy flux conservative during the extrapolation. We call this propagator the transparent propagator (Wu and Cao, 2005).
Asymptotic true-amplitude one-way propagators

Theoretically there is no approximation when transforming the original form of the asymptotic true-amplitude one-way propagator (4) into the energy-flux conserved propagator (12). However, for the implementation of transforming back to the pressure from flux, we need to expand the fourth root operator $\Lambda^{1/2}$ with a rational function approximation. This approximation may introduce a numerical error to the amplitude of the wide-angle waves, which will be shown in a numerical example below. Therefore, in the following examples, we utilize the above-mentioned “correction at every extrapolation step” except when otherwise specified.

The first-order Padé approximation for $\Lambda^{1/2}$ can be written as

$$\Lambda^{1/2} = \sqrt{k(x, z)} \left[ 1 + \frac{a_1}{k(x, z)} \frac{\partial^2}{\partial x^2} + \frac{b_2}{k(x, z)} \frac{\partial^2}{\partial x^2} \right]^{-1/2},$$

with $a_2=0.25$, $b_2=0.375$.

Numerical results

For simplicity, numerical examples shown here are for 2D models. It is straightforward to extend them to the 3D case. The model dimension is 10.24 km $\times$ 5.12 km. The source is located at $x_0=5.12$ km on the surface. The peak frequency of the source is 15 Hz. To clearly compare the wavefield amplitude, we plot the amplitude versus distance from the source along radial directions for different angles $\theta$ (see Figure 1). We will compare the one-way propagator amplitude with that obtained from the full-wave finite-difference method, which is considered as the true solution in general heterogeneous media.

![Figure 1: Diagram for extracting wavefield amplitude along the radial directions for different angles ($\theta$). The star and dots represent the location of the source and receivers respectively.](image)

Effect of lateral velocity variation on the amplitude

To show the effect of the lateral velocity variation related correction term on the amplitude in the asymptotic true-amplitude one-way propagator (4), we first use a model with only lateral velocity variation: $v(x)=3.0+0.5\sin(2\pi(x/x_0-1/4))$ (km/s). Without the lateral variation term, the conventional one-way propagator cannot yield accurate amplitudes for angles larger than $0^\circ$ (Figure 2a). The larger the angle is, the larger the amplitude error is. For $30^\circ$, the amplitude error at 4 km is about 15%. With that term, the obtained amplitude agrees very well with the true amplitude for angles up to $30^\circ$ (Figure 2b). The amplitude error for larger angles is also smaller than that without the term. Next we introduce a linear velocity variation along the vertical direction to the above model: $v(x, z)=3.0+0.5\sin(2\pi(x/x_0-1/4))+0.36z$ km/s. Results show that the one-way propagator without the lateral velocity variation term cannot yield accurate amplitudes for angles larger than $0^\circ$ even with the correction term $\Gamma$ (Figure 3b). Comparison of Figure 3a and Figure 3c shows that the lateral velocity variation related correction term in $\Lambda$ has a significant contribution to the amplitude. With both the lateral and vertical velocity variation related corrections, the true-amplitude one-way propagator yields accurate amplitudes for angles up to $30^\circ$ in this model (Figure 3d).
amplitude accuracy for the large angle waves can be improved by optimizing the expansion coefficients in the approximate asymptotic true-amplitude one-way propagator, as shown later.

**Flux transparent propagator**

In the previous numerical examples, we applied the amplitude correction term \( \Gamma \) at every extrapolation step. Here we show the results from the flux transparent propagator. We use the same \( v(x, z) \) model as above. Results show that the amplitudes obtained from the flux transparent propagator (without/with the lateral velocity variation correction; Figure 4a, b) are indistinguishable from those obtained by applying the \( \Gamma \) correction at every step (Figure 3b, d) for smaller angles (e.g., 0° to 45°). However, for larger angles (e.g., 75°) the amplitude error for the flux propagator is larger than that by applying the correction at every step. This is consistent with previous theoretical analysis.

**Optimized asymptotic true-amplitude one-way propagator**

To achieve better accuracy for the wide-angle waves, one approach is to use a higher-order expansion; however, this can bring a lot of extra computation. Here we adopt the other approach, optimizing the expansion coefficients in the approximated propagators. We optimize the expansion coefficients with a global optimization scheme similar to that in Huang and Fehler (2000) but with an extra parameter related to the lateral velocity variation. The optimized coefficients are obtained by a search algorithm that maximizes the propagation angle at a given expansion error for all model parameters. With the optimized expansion coefficients in the wide-angle screen type true-amplitude one-way propagator \( (11) \), both the amplitude (Figure 5) and travel time (Figure 6) for wide-angle waves have been improved compared with the amplitude (Figure 2b and Figure 3d) and phase using the original coefficients.

**Conclusions**

We demonstrate that lateral velocity variation can have a significant effect on the amplitude of the asymptotic true-amplitude one-way propagator. We derive a dual-domain wide-angle screen type asymptotic true-amplitude one-way propagator to preserve the correct amplitude of one-way generalized screen propagators. By theoretical and numerical analysis, we evaluate two implementations of the amplitude correction: correction at every extrapolation step, and a flux transparent propagator. In the latter implementation, the numerical conversion from flux to pressure may introduce numerical error to the amplitude for wide-angle waves, but it is more efficient for target-oriented applications. Optimization of the expansion coefficients in the asymptotic true-amplitude one-way propagator can improve both the amplitude and phase accuracy for wide-angle waves.

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