Wave equation based illumination analysis
Xiao-Bi Xie* IGPP, University of California, Santa Cruz, Shengwen Jin, Screen Imaging Technology Inc., and Ru-Shan Wu, IGPP, University of California, Santa Cruz

Summary

In this research, the formulations for wave equation based illumination analysis are presented. Different types of illuminations that can be extracted from this approach are investigated. Using numerical examples, we discussed physical meanings and potential applications of these illumination measurements.

Introduction

The angle-domain information plays an important role in seismic data processing. This information is widely used in seismic velocity updating, illumination analysis and AVA analysis, etc. Traditionally, angle related analysis is conducted using ray based methods (Bear et al., 2000; Muerdter et al., 2001abc; Xu et al., 2001). The wave equation based approaches usually do not generate the directional information, which prevents them from being directly used for observations requiring localized angle information. Recently, we developed techniques to decompose the wavefield and obtain the localized angle-domain information. Two methods are adopted for this purpose, the local plane wave analysis technique (Xie and Wu, 2002) and a beamlet domain propagator based on Gabor-Daubechies frame (Wu et al., 2000, Wu and Chen 2001). Both methods decompose the wavefield into localized beamlets carrying angle information. Using these techniques, Xie and Wu (2002) calculated angle domain common image gathers. Wu and Chen (2002, 2003) and Wu, et al. (2003) calculated reflector dip-response for a given acquisition geometry and velocity model. Xie, et al. (2003) developed a wave equation based illumination technique, which provided full volume illumination analysis in complicated 3D models.

In this research, we will present various types of illumination measurements that can be extracted from the wave equation based approach. We first briefly give the formulation of wave equation based illumination analysis. Then, discuss illumination measurements in different domains based on the formulation. A set of numerical examples are presented to explain the calculation of these illumination measurements, followed by discussions of their potential applications.

Formulation

Consider using a survey system composed of a source located at \( r_s \) and a receiver located at \( r_g \) to investigate the subsurface target region \( V \) which surrounds \( r \) (see Figure 1). The reflected (scattering) wave from the target region which reaches to the receiver can be expressed as

\[
\begin{align*}
    u(r, r_g) &= \int_V m(r') G(r', r) G(r', r_g) \, dk' \\
    &= \int_V m(r') G(r', r_g) G(r, r_g) \, dk' \\
    &= \int_V m(r') G(r', r_g) G(r_s, r_g) \, dk' \\
    &= \int_V m(r') e^{i k_s \cdot r_g} - k_s \cdot r_g \, dk' \\
    &= \int_V m(r') e^{i (k_s + k_g) \cdot r_g} \, dk' \\
\end{align*}
\]

Figure 1. Sketch showing the coordinate system.

where \( m(r') = -k_0^2 c_0^2 / c^2(r') - 1 \), \( c(r') \) is the velocity, \( c_0 \) is the reference velocity inside \( V \), \( k_0 = \omega c_0 \), \( G(r', r_g) \) and \( G(r_s, r_g) \) are Green’s functions propagating waves from \( r_s \) to \( r' \) and from \( r_g \) to \( r' \). The reciprocity \( G(r', r_g) = G(r_g, r') \) is used here. For these Green’s functions, we will use accurate one way propagators (e.g., Stoffa et al., 1990; Ristow and Ruhl, 1994; Jin, et al., 2002; Xie and Wu, 1998; Huang and Fehler, 2000; Xie et al., 2000; Wu and Chen, 2001; Biondi, 2002). Within the small target region \( V \), the local Born approximation can be adopted. Introducing the local plane wave decomposition within \( V \)

\[
    G(r', r_g) = \int G(k_r, r_g) e^{i k_r \cdot r_g} \, dk \\
    G(r, r_g) = \int G(k_r, r_g) e^{i k_r \cdot r_g} \, dk \\
\]

we have

\[
    u(r, r_g) = \int G(k_r, r_g) G(k_r, r_g) \times \\
    \times m(r, k_r + k_g) \, dk_g \\
    = \int V m(r') e^{i (k_s + k_g) \cdot r_g} \, dk' \\
\]

where

\[
    m(r, k_r + k_g) = \int_V m(r') e^{i k_r \cdot r_g} \, dk' \\
\]
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In these equations, \( \mathbf{k}_s \) and \( \mathbf{k}_r \) are local transforms with respect to \( \mathbf{r}' \) (not \( \mathbf{r}_s \) or \( \mathbf{r}_r \)). Subscripts \( s \) and \( g \) denote the waves coming from the source and receiver, respectively. We retain the parameter \( r \) to indicate the location of the target region. Equation (3) links the observation and the source with the subsurface target. It forms the basis of many seismic methods for recovering the subsurface target parameters (e.g., seismic migration, inversion and tomography). From another point of view, equation (3) also tells us, given a velocity model, how well the target at \( \mathbf{r} \) can be “illuminated” by the survey configuration at \( (\mathbf{r}_s, \mathbf{r}_r) \). The integrand in equation (3) describes a scattering event which is composed of an incident wave in the \( \mathbf{k}_s \) direction and a scattered wave leaving the target in the \( \mathbf{k}_r \) direction. It appears that, in the local wavenumber domain, the target is illuminated by the factor \( G(\mathbf{k}_s, \mathbf{r}_s; \mathbf{k}_r, \mathbf{r}_r) \). To simulate the actual process expressed in equation (3), we define a function to describe the target illumination. To allow easier stacking, we choose energy, instead of the amplitude, in our equations. The illumination \( D(\mathbf{r}, \mathbf{r}_s, \mathbf{r}_r) \) can be written as

\[
D(\mathbf{r}, \mathbf{r}_s, \mathbf{r}_r) = \int [l(\mathbf{k}_s, \mathbf{r}_s; \mathbf{k}_r, \mathbf{r}_r)] [G(\mathbf{k}_s, \mathbf{r}_s; \mathbf{k}_r, \mathbf{r}_r)] \times M(\mathbf{r}; \mathbf{k}_s + \mathbf{k}_r)/k_s dk_s,
\]

where

\[
l(\mathbf{r}, \mathbf{k}_s; \mathbf{r}_s) = G(\mathbf{r}, \mathbf{k}_s; \mathbf{r}_s) \times G(\mathbf{r}, \mathbf{k}_r; \mathbf{r}_r)
\]

are expressions for the energy fluxes from the source and receiver to the target region, respectively. \( M(\mathbf{r}; \mathbf{k}_s + \mathbf{k}_r) \) is a characteristic function of the local structure, which keeps the mapping relationship between the model and the incoming/scattering waves but gives unit scattering amplitude. We define

\[
A(\mathbf{r}, \mathbf{k}_s, \mathbf{k}_r; \mathbf{r}_s, \mathbf{r}_r) = l(\mathbf{r}, \mathbf{k}_s; \mathbf{r}_s) / l(\mathbf{r}, \mathbf{k}_r; \mathbf{r}_r)
\]

as the local illumination matrix of the source-receiver pair \((\mathbf{r}_s, \mathbf{r}_r)\). Equation (5) becomes

\[
D(\mathbf{r}, \mathbf{r}_s, \mathbf{r}_r) = \int [A(\mathbf{r}, \mathbf{k}_s, \mathbf{k}_r; \mathbf{r}_s, \mathbf{r}_r)] \times M(\mathbf{r}; \mathbf{k}_s + \mathbf{k}_r)/k_s dk_s
\]

For a system composed of multiple sources and receivers, the illumination can be written as

\[
D(\mathbf{r}) = \sum_{\mathbf{r}_s, \mathbf{r}_r} D(\mathbf{r}, \mathbf{r}_s, \mathbf{r}_r)
\]

By substituting (9) into (10)

\[
D(\mathbf{r}) = \int \left[ \sum_{\mathbf{r}_s, \mathbf{r}_r} A(\mathbf{r}, \mathbf{k}_s, \mathbf{k}_r; \mathbf{r}_s, \mathbf{r}_r) \right] M(\mathbf{r}; \mathbf{k}_s + \mathbf{k}_r)/k_s dk_s
\]

where

\[
A(\mathbf{r}, \mathbf{k}_s, \mathbf{k}_r) = \sum_{\mathbf{r}_s, \mathbf{r}_r} A(\mathbf{r}, \mathbf{k}_s, \mathbf{k}_r; \mathbf{r}_s, \mathbf{r}_r)
\]

is the illumination matrix of the entire survey system, which is composed of all possible scattering events from different directions (Wu and Chen, 2002; Xie et al., 2003). For a dipping structure given by \( M(\mathbf{r}) \sim \delta(\mathbf{r} - \mathbf{n}) \), we have \( M(\mathbf{k}) - \delta(\mathbf{k} - \mathbf{c} \mathbf{n}) \). This gives \( \mathbf{k}_s + \mathbf{k}_r - \mathbf{c} \mathbf{n} = 0 \) with \( \mathbf{n} \) as the normal vector of the dipping structure. \( \mathbf{n} \) is also the angle bisector of \( \mathbf{k}_s \) and \( \mathbf{k}_r \) (see Figure 1). Since \( \mathbf{k}_n \mid = 0 \), \( c = 2k_g \cdot \mathbf{n} = 2k_s \cdot \mathbf{n} = 2k_0 \cos i \cdot i \) is the incident-reflection angle. Using this \( M \) in equation (11) gives

\[
D(\mathbf{r}, \mathbf{n}) = \int [A(\mathbf{r}; \mathbf{k}_s, \mathbf{k}_r)] \delta(\mathbf{k}_s + \mathbf{k}_r - \mathbf{c} \mathbf{n})/k_s dk_s
\]

where

\[
\mathbf{k}_d = \mathbf{k}_s + \mathbf{k}_r, \quad \mathbf{k}_c = \mathbf{k}_s - \mathbf{k}_r
\]

Given a velocity model and acquisition configuration, equation (13) gives the ADR (acquisition dip response) at target location \( \mathbf{r} \) for dipping \( \mathbf{n} \). The scattering events that actually contribute to the illumination of a dipping structure are distributed through out a plane within the 6D space \( (\mathbf{k}_s, \mathbf{k}_r) \). They are picked by \( M(\mathbf{k}) \). Using the coordinate rotation \( \mathbf{k}_d = \mathbf{k}_s + \mathbf{k}_r, \quad \mathbf{k}_c = \mathbf{k}_s - \mathbf{k}_r \)

(14)

we have

\[
D(\mathbf{r}, \mathbf{n}) = \int [A(\mathbf{r}; \mathbf{k}_s, \mathbf{k}_r)] \delta(\mathbf{k}_r)/k_s dk_s
\]

(15)

Under the new coordinate system, \( \mathbf{k}_d \) links to the dipping angle and \( \mathbf{k}_c \) is related to the reflection angle \( i \). Considering an isotropic point scatterer \( M(\mathbf{r}) \sim \delta(\mathbf{r}) \), we have

\[
M(\mathbf{k}) - \mathbf{c} \mathbf{n} = 0\]

(16)

Equation (16) is the target total illumination which sums energy from all possible scattering combinations. Other models may also be considered, such as the reflector with a specific curvature.

Numerical issues

For a 3D velocity model, the integrals in the above equations are carried out in a 6D space \( (\mathbf{k}_s, \mathbf{k}_r) \) or \( (\mathbf{k}_d, \mathbf{k}_c) \). Since \( k^2 + k^2 + k^2 = k^2 \), the wavenumber \( \mathbf{k} \) can be uniquely determined by its horizontal component \( \mathbf{k}_T = \mathbf{k} - \mathbf{k}_c \bar{\mathbf{e}}_z \), or equivalently by a direction (angle) \( \theta = (\theta, \varphi) \). By changing the variables to \( \mathbf{k}_T \) or \( \theta \), the integrations can be reduced to 4D. Proper Jacobians should be included in the transform. In practical application, the problem can be discretized. Function \( A \) becomes a matrix and the integrals over \( \mathbf{k} \) are replaced by summations. This is the reason we call \( A \) the local illumination matrix. For a 2D velocity model and using \( \mathbf{k}_d \) or \( \theta \), the local illumination matrix is two-dimensional.

The size of the target region \( V \) used for the local plane wave decomposition should be small enough to keep the localized properties of both the wavefield and the model, but also big enough to preserve the internal structures of the wavefield and the model (the wave propagation direction and the structure dipping, etc.). The Heisenberg uncertainty rule should be followed. The local plane wave decomposition using equation (2) can be easily conducted in the \( \mathbf{k}_T \) domain when combined with the one-way propagator and windowed FFT (Xie and Wu, 1992; Xie et al., 2003).
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Calculations of different illumination measurements

To demonstrate the potential applications of the wave equation based illumination approach, we calculated different types of illumination measurements.

(1) Local illumination matrix. The local illumination matrix can be calculated using equations (8) or (12). As an example, Figure 2 gives local illumination matrixes at different locations in the 2D SEG/EAGE salt model.

Figure 2. Local illumination matrix in SEG salt model.

(2) Illumination angle distribution. The partially integrated function \( A(\mathbf{r}, k, c\mathbf{n}(\mathbf{r}))|_{\text{target}} \) in equation (15) gives the illumination as a function of reflection angle for a structure with dipping \( \mathbf{n} \), where \( \mathbf{n} = \mathbf{n}(\mathbf{r}) \) is the normal vector along the target horizon. Figure 3 gives the angle distribution of these illuminations along four target horizons in a constant velocity model. Even for such a simple model, the result reveals the joint effect of acquisition aperture and target dipping on the illumination.

(3) Target oriented illumination. The acquisition dip response on a target can be calculated using equation (13) or (15) with \( D(\mathbf{r}, \mathbf{n}(\mathbf{r}))|_{\text{target}} \). Figure 4 gives the ADR for the same four targets in Figure (3). It clearly reveals the amplitude changes caused by the acquisition aperture and target dip angle. The minimum illumination corresponds to sections with steep dip angles.

Figure 3. Illumination angle distribution on targets.

Figure 4. Acquisition dip response on targets.

(4) Wavenumber domain illumination. As mentioned in equations (4) and (14), \( \mathbf{k} \) is related to the wavenumber of the model. Projecting the illumination matrix \( A \) onto \( \mathbf{k} \), reveals the wavenumber domain illumination of the structure. Similarly, mapping \( A \) onto \( \mathbf{k} \), reveals the reflection angle coverage. The resolution power of the migration/inversion and tomographic methods in the wavenumber domain can be obtained from this information (Wu and Toksöz, 1987; Woodward, 1992). Figure 5 shows examples of such illuminations. Figure 6 illustrates the wavenumber domain illumination at selected locations in the SEG/EAGE salt model.

Figure 5. Wavenumber domain illumination in \( k_y \) domain (left) and \( k_x \) domain (right) for a 2D model.

Figure 6. Wavenumber domain illumination at selected locations in the SEG/EAGE salt model.

(5) Volume illumination analysis. Volume ADR maps in space domain can be calculated using equations (13) or (15). Figure 7 gives a vertical profile of the vertical ADR in...
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the 3D SEG/EAGE salt model. The volume total illumination can be calculated using equation (16).

Figure 7. 3D SEG/EAGE salt model (upper panel) and vertical acquisition dip response (lower panel).

Conclusions
The local plane wave decomposition method and wavelet transform method provide the basis to investigate the wave propagation in angle domain. Based on the angle domain information, we formulated the illumination equations. Different types of illumination measurements can be extracted from these equations. They can be used to evaluate the detecting power of the acquisition system to the subsurface target region. Different types of illuminations were presented using numerical examples and their potential applications were discussed.

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References


