Wide-angle Thin-slab Propagator with Phase Matching for Elastic Wave Modeling
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Summary
The effect of the local Born approximation on phase delay or travel time is analyzed. The wide-angle thin-slab propagator of elastic wave given by Wu (1994) has limited accuracy for phase change due to propagation. A phase matching method is introduced to the wide-angle thin-slab propagator for elastic waves. Synthetic seismograms for the French model are calculated using the thin-slab propagator with phase matching and good agreement with the finite-difference results is obtained.

Introduction
Based on the MFSB (multiple foreshocking single backscattering) approximation of De Wolf (1985), one-way wave equations for elastic waves were derived by Wu (1994). The local Born approximation is used within a thin slab in the derivation. However, numerical experiments show that the accumulated effects of phase delays in the forward direction due to Born approximation are strong although the effects within a thin slab are small. In this paper the effect of the local Born approximation on phase delay or travel time is analyzed. A phase matching method similar to the phase-screen approach for scalar waves (Wu and Huang, 1995; Wu and de Hoop, 1996) is introduced and applied to the wide-angle thin-slab propagator for elastic waves. Synthetic seismograms are calculated using the thin-slab propagators with or without phase matching and compared with the results of a finite-difference method.

Thin-slab propagator
Using the elastic wave Rayleigh integrals and local elastic Born scattering theory, Wu (1994) derived the wide-angle thin-slab propagator for calculating forward and backward scattered fields by a heterogeneous thin slab. In this study, only P-P scattered field is concerned.

\[ -k_\alpha(k_\alpha, \tilde{k}_\alpha) \int dx e^{-i k_x x} \frac{2 \delta \mu(x, z)}{\lambda_0 + 2 \mu_0 k_\alpha^2} \epsilon(x, z) \]

(1)

With \( k_x^2 = + \gamma_\alpha \) for foreshocking and \( k_x^2 = - \gamma_\alpha \) for backscattering. In (1) \( u_0^\alpha(x, z) \) (displacement), \( \Delta \cdot u_0^\alpha(x, z) \) and \( \epsilon_0^\alpha(x, z) \) (strain) can be calculated by

\[ u_0^\alpha(x, z) = \frac{1}{2 \pi} \int dk_x e^{i k_x x} u_0^\alpha(k_x) e^{i \gamma_\alpha x} \]

(2)

\[ \nabla \cdot u_0^\alpha(x, z) = \frac{i k_\alpha^0}{2 \pi} \int dk_x e^{i k_x x} k_{\alpha} \cdot u_0^\alpha(k_x) e^{i \gamma_\alpha x} \]

(3)

\[ \epsilon_0^\alpha(x, z) = \frac{i k_\alpha^0}{2 \pi} \int dk_x e^{i k_x x} k_{\alpha} \cdot u_0^\alpha(k_x) e^{i \gamma_\alpha x} \]

(4)

where \( \gamma_\alpha = (\delta \mu^2 - k_x^2)^{1/2} \); \( \gamma_\alpha' = (k_\alpha^2 - k_x^2)^{1/2} \)

\( k_\alpha^0 = \omega / \alpha_0 \) is P wavenumber, \( \alpha_0 \) is P-wave velocity in the background medium. The unit wavenumber vectors \( k_{\alpha} \) and \( k_{\alpha}^0 \) are

\[ k_{\alpha} = \frac{1}{k_\alpha^0} (k_x, \gamma_\alpha) \]

\[ k_{\alpha}^0 = \frac{1}{k_\alpha^0} (k_x, \gamma_\alpha') \]

\( u_0^\alpha(k_x) \) is the incident P-wave displacement at the entrance \( x = z' \). \( z = z_1 \) is the thin-slab exit. The slab thickness is \( \Delta z = z_1 - z' \).

Phase matching for forward scattering
To mathematically estimate the phase delay by equation (1), we assume that the thin slab has constant perturbations for P- and S-wave velocities and density (see Figure 1), i.e., a homogeneous thin slab. Substituting (2-4) into (1), the P-P foreshattered field

\[ U^{PP}(k_x, z') = \frac{i k_\alpha^0}{2 \gamma_\alpha} \int dx e^{-i k_x x} \frac{\delta \rho(x, z)}{\rho_0} u_0^\alpha(x, z) \]

\[ \tilde{k}_\alpha \cdot \int dx e^{-i k_x x} \frac{\delta \lambda(x, z)}{\lambda_0 + 2 \mu_0 k_\alpha^2} u_0^\alpha(x, z) \]

\[ -k_\alpha \int dx e^{-i k_x x} \frac{\delta \mu(x, z)}{\lambda_0 + 2 \mu_0 k_\alpha^2} \nabla \cdot u_0^\alpha(x, z) \]

(5)

Figure 1: A thin slab model. \( z' \) is the entrance plane and \( z_1 \) is the exit plane. \( \rho_0, \lambda_0 \) and \( \mu_0 \) are the parameters of the background medium and \( \rho, \lambda \) and \( \mu \) are the parameters of the thin slab.

at \( z = z_1 \) plane can be obtained

\[ U^{PP}(k_x, z_1) = \frac{i k_\alpha^0}{2 \gamma_\alpha} \int dx e^{-i k_x x} \frac{\delta \rho}{\rho_0} \frac{\delta \lambda + 2 \delta \mu}{\lambda_0 + 2 \mu_0} c^{i \gamma_\alpha \Delta z} u_0^\alpha(k_x, z') \]
Thin-slab propagator with phase matching

In order to obtain optimum phase match in normal direction for backscattering, assume that a plane P-wave is incident perpendicular to the thin slab. For this case we have \( k_x = 0 \) and \( \gamma_\alpha = k_\alpha^0 \). The total outgoing P-wave at the thin-slab exit (\( z = z_1 \)) can be expressed by

\[
u_\alpha(0,z_1) = e^{i k_\alpha^0 z_1} u_\alpha^0(0,z_1') + U^{P'}(0,z_1')
\]

\[
= \left[ 1 + \frac{i k_\alpha^0}{2} \left( \frac{\delta \rho}{\rho_0} - \frac{\delta \lambda + 2\delta \mu}{\lambda_0 + 2\mu_0} \right) \right] \Delta z e^{i k_\alpha^0 z_1} u_\alpha^0(0,z_1')
\]

(6)

The first term in (6) is the background field propagating in the background medium. Using the Rytov approximation, the phase delay can be obtained

\[
\phi_{\text{slab}} = k_\alpha^0 \left[ 1 + \frac{1}{2} \left( \frac{\delta \rho}{\rho_0} - \frac{\delta \lambda + 2\delta \mu}{\lambda_0 + 2\mu_0} \right) \right] \Delta z
\]

(7)

Equation (7) results from the local Born approximation. We rewrite the perturbation factors in (7) into

\[
\frac{\delta \rho}{\rho_0} - \frac{\delta \lambda + 2\delta \mu}{\lambda_0 + 2\mu_0} = \frac{\delta \rho}{\rho} - \frac{\delta \lambda + 2\delta \mu}{\lambda + 2\mu} + \left( \frac{\delta \rho}{\rho_0} - \frac{\delta \rho}{\rho} \right) - \left( \frac{\delta \lambda + 2\delta \mu}{\lambda_0 + 2\mu_0} - \frac{\delta \lambda + 2\delta \mu}{\lambda + 2\mu} \right)
\]

(8)

For weak heterogeneities, we have the following approximation

\[
\frac{\delta \rho}{\rho} - \frac{\delta \lambda + 2\delta \mu}{\lambda + 2\mu} \approx \frac{-2\delta \alpha}{\alpha}
\]

(9)

Substituting (8) and (9) into (7) and notice that \( k_\alpha^0 (1 - \frac{k_x^0}{k_\alpha^0}) = k_\alpha \), the phase delay can be expressed by

\[
\phi_{\text{slab}} = k_\alpha^0 \left[ 1 + \frac{k_\alpha^0}{2} \left( \frac{\delta \rho}{\rho_0} - \frac{\delta \rho}{\rho} \right) - \left( \frac{\delta \lambda + 2\delta \mu}{\lambda_0 + 2\mu_0} - \frac{\delta \lambda + 2\delta \mu}{\lambda + 2\mu} \right) \right]
\]

(10)

where \( k_\alpha = \frac{\gamma_\alpha}{\alpha} \), \( \alpha \) is P-wave velocity in the thin slab.

From (10) we see that the first term gives the exact phase delay in z-direction and other terms give the phase-delay errors. From (10) we also see that if the perturbations \( \frac{\delta \rho}{\rho_0}, \frac{\delta \lambda}{\lambda_0 + 2\mu_0} \) and \( \frac{\delta \lambda}{\lambda_0 + 2\mu_0}, \frac{\delta \mu}{\lambda_0 + 2\mu_0} \) in (1) are replaced by \( \frac{\delta \rho}{\rho}, \frac{\delta \lambda}{\lambda + 2\mu} \) and \( \frac{\delta \lambda}{\lambda + 2\mu}, \frac{\delta \mu}{\lambda + 2\mu} \), respectively, the phase delay given by (1) will be exact in z-direction, i.e., the optimum phase match. Following the same steps, a similar result for S-S scattering can be also obtained.

Figure 2 shows P-wave (a) and S-wave (b) phase delays for the propagators with/without phase matching versus perturbations. Suppose that a plane P- or S-wave propagates perpendicular to a homogeneous slab which can be sliced into 40 homogeneous thin slabs with thickness of 8m. Phase delays can be directly calculated using (1) and its modified version with phase matching, respectively. The perturbations of P- and S-wave velocities range from -30% to 30%. The phase delays from the ray method (dashed lines) are also drawn in Figure 2. From Figure 2 we see that the thin-slab propagator without phase matching gives a good approximation only if velocity perturbations are less than 5%. While the thin-slab propagator with phase matching almost gives the exact travel time even for high perturbation cases.

**Figure 2:** Phase delays of P-wave (a) and S-wave (b) versus perturbations. In this figure planar P- and S-waves are incident perpendicular to the thin slab. The solid lines are from the thin-slab propagator with or without phase matching and the dashed lines are from ray theory. The frequency is 20Hz.

**Angle dependence of accuracy**

In the above section the phase matching method for elastic waves is introduced. The phase delays of the elastic waves propagating perpendicular to the thin slab are corrected exactly. Under the approximation of weak constant heterogeneities and small angle scattering, for a plane P-wave incident upon the thin slab at an arbitrary angle, the residual of the phase delays
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between the wide-angle thin-slab propagator and high frequency approximation (ray method) can be analytically derived.

\[
\Delta \phi_{\text{with}} = \frac{1}{2} \left| \frac{\delta \alpha}{\alpha_0} \right|^2 \frac{k_2^2}{\gamma_0} \Delta z + k_x \Delta z' \quad (11)
\]

with phase matching, and

\[
\Delta \phi_{\text{without}} = \frac{1}{2} \left| \frac{\delta \alpha}{\alpha_0} \right|^2 \frac{k_0^2}{\gamma_0} \Delta z + k_x \Delta z' - \frac{k_0^0}{\gamma_0} \left| \frac{\delta \alpha}{\alpha_0} \right|^2 k_x \Delta z \quad (12)
\]

without phase matching. Where \( \Delta z' \) is the difference of horizontal propagation distances between one-way method and ray method and a higher order small quantity. Figure 3 shows the errors of phase delays of the wide-angle thin-slab propagators with (upper curve) or without (lower one) phase matching and ray method.

Figure 3: The residual of phase delays between the wide-angle thin-slab propagators with (upper curve) or without (lower one) phase matching to ray method versus incident angle. From Figure 3, for small angle scattering, the effect of the phase matching on phase delay is important but the effect of incident angle is not significant.

Application

Synthetic seismograms are conducted to demonstrate the improvement. Figure 4 shows a 2D physical model which is a slice cut from the French model (1974). The parameters of the background medium are \( \alpha_0 = 3.6 \text{km/s} \), \( \beta_0 = 2.68 \text{km/s} \) and \( \rho_0 = 2.2 \text{g/cm}^3 \). The intermediate layer has -20% perturbations for both P- and S-wave velocities. Source and receiver geometry is also shown in Figure 4. A Ricker wavelet with center frequency of 20Hz is adopted in this study.

The synthetic seismograms are calculated using the thin-slab propagators with and without phase matching, respectively, and compared with the results calculated using a fourth-order elastic wave finite-difference code (Xie and Yao, 1988). The one-way wave method has model parameters of \( dx = 8m \) and \( \Delta z = 8m \). The 2D finite-difference method has parameters of \( dx = dz = 8m \) and \( dt = 0.001s \). The direct arrivals have been properly removed from those results obtained using finite-difference method. A comparison between the thin-slab propagator without phase matching and the finite-difference method is shown in Figure 5. (a) and (b) for horizontal and vertical components of displacement, respectively. The solid lines are from the finite-difference method and the dashed lines are using the thin-slab propagator. Events ①, ② and ③ are the reflections from the upper irregular interface. Events ④ and ⑤ are the reflections from the lower horizontal interface. From Figure 5a and 5b it is very clear that the reflections caused by the upper interface have good matches between the thin-slab propagator method and the finite-difference method, even for those receivers which are far from the source, while the reflections caused by the lower interface do not match well, especially for travel times. This is because those reflections are related to the foreshadowing of the thin-slab propagator in which the phase delay compensation is not matched. Figure 6 shows a comparison between the thin-slab propagator with phase matching and the finite-difference method. Compared with Figure 5, events ④ and ⑤ in Figure 6 are greatly improved. Both amplitudes and arrival times are in good agreement with the finite-difference method.

Conclusions

We applied a phase matching method similar to the phase-screen approach for scalar waves, to thin-slab propagator given by Wu (1994), which can give more accurate phase delay than the original version. Application examples in this paper show that the synthetic seismograms calculated with the phase matched thin-slab propagator are in good agreement with the results from the finite-difference method.
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Figure 5: Synthetic seismograms from different methods. The solid lines are from the finite-difference method and the dashed lines are calculated using the thin-slab propagator given by Wu (1994). (a) and (b) are horizontal and vertical components of displacement, respectively.

Figure 6: Synthetic seismograms from different methods. The solid lines are from the finite-difference method and the dashed lines are calculated using the thin-slab propagators with phase matching.

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References

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