Infinite element-based absorbing boundary technique for elastic wave modeling
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Summary

An absorbing boundary scheme using an infinite element algorithm has been given for elastic wave modeling by solving the boundary (or volume) integral equations related to the free space Green’s functions. The integral equations have the ability to include infinite domains. We place an infinite element at the end point of the boundary extending to infinity instead of the artificial boundary at the edge of domain of computation. Then infinite shape functions are constructed for the infinite element. The unknown (displacement or stress) in the infinite element is assumed to vary in the infinite direction from the values at the end point according to decay functions. The absorbing boundary scheme do not require the computation of incident angles, the separation of vector fields into compressional and shear components, and the assumption of elastic medium being homogeneous in the region adjacent to the boundary. Real examples illustrate the effectiveness of the infinite element-based absorbing boundary.

Introduction

With the aid of the free space Green’s functions, the partial differential equation for elastic waves can be transformed into the boundary integral (BI) equation for homogeneous medium or the volume integral (VI) equation for inhomogeneous medium. The former can be solved effectively by the boundary element (BE) method (Fu and Mu, 1994), and the latter solved via the velocity-weighted wavefield (Fu et al., 1997). However, in these calculations of wave propagation, artificial reflections arise at the edges of domain of computation. To avoid these spurious reflections, a simple method is to enlarge the domain of computation so that reflected signals can not reach the interesting region within the given time period. Obviously, the approach wastes computational resources, particularly for three-dimensional (3-D) computations.

Alternatively, the absorbing boundary conditions for finite-difference calculations may be employed to reduce the amplitude of the reflected waves, but sometimes can not produce perfect results when applied to solutions of the BI or VI equation for modeling of elastic waves. The most widely used schemes for elastic waves appear to be those based on paraxial approximations to the elastic wave equation. The paraxial boundary conditions (Clayton and Engquist, 1977; Engquist and Majda, 1979) is easily implemented, but perfectly absorb the elastic waves only for waves nearly normally incident upon the boundary. Very high reflectivities appear for waves traveling obliquely to the boundary. The boundary conditions based on the schemes of Lindman (1975) and Randall (1988) can absorb obliquely incident waves to a great extent, but is not convenient to be used in solutions of the BI or VI equation, particularly for the VI equation where the unknown is only the displacement, without the normal gradient or stress involved. Some limitations and assumptions are necessary for the above absorbing boundary conditions. For instance, elastic waves are processed by the separation of the incident vector displacement fields into compressional and shear components. The elastic medium is assumed to be homogeneous in the region adjacent to the artificial boundary. The performance of the boundary conditions depends on the Poisson’s ratio of the elastic medium.

In this paper, an efficient absorbing boundary scheme based on an infinite element technique is described for calculations of the solutions to the BI or VI equation for elastic wave propagation. The approach overcomes some of the difficulties of the above absorbing boundary conditions, and takes less memory space and computing time.

Lindman’s absorbing boundary for the BI equation

Consider the below elastic equation for steady state wave in a homogeneous domain Ω bounded by a closed surface Γ,

\[ \mu \nabla^2 u(r) + (\lambda + \mu) \nabla \cdot u(r) + \rho \omega^2 u(r) = -f(r). \]  

(1)

Suppose the source distribution is focused on the point s. With the aid of free space Green’s functions, equation (1) can be transformed equivalently into the following boundary integral equation for the displacement components \( u_j(r) \) at a location \( r \) on the boundary

\[ C_g(r) u_j(r) + \int_{\Gamma} T_j(r, r') u_j(r') d\Gamma(r') = \int_{\Gamma} U_j(r, r') t_j(r') d\Gamma(r') + \rho U_j(r, s) f_j(\omega), \quad (i, j = 1, 2, 3), \]  

(2)

where the coefficients \( C_g(r) \) generally depend on the local geometry at \( r \), \( t_j(r') \) are the stress components, \( U_j(r, r') \) and \( T_j(r, r') \) are the fundamental solutions.

For the domain extending infinity, an artificial boundary \( \Gamma_\infty \) is placed at the edges of domain of computation to form a closed surface, and the boundary integrals on \( \Gamma_\infty \) are assumed to vanish, that is,
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\[ \lim_{r \to \infty} \int_{\Gamma_a} (U_i t_j - \Gamma_i u_j) d\Gamma = 0. \]  
(3)

Using approximate relations in the fundamental solutions as \( r = |r - r'| \) approaches infinity, we have the scalar absorbing boundary condition denoted by

\[ \frac{\partial u}{\partial n} = (ik \cos \theta) u. \]  
(4)

where \( \theta \) is the incident angle measured from the normal to the boundary, \( k \) is the scalar wavenumber, and \( \partial / \partial n \) denotes differentiation with respect to the outward normal to \( \Gamma_a \), and also have the elastic absorbing boundary condition expressed as

\[ \lim_{r \to \infty} r (iapV_r \cos \theta u - t) \cdot r = 0 \]
\[ \lim_{r \to \infty} r (iapV_r \sin \theta u - t) \cdot \hat{r} = 0 \]  
(5)

where \( \hat{r} \) is perpendicular to \( r \). Let \( k_p = iapV_r \cos \theta \), \( k_s = iapV_s \cos \theta \), \( r = (m,n) \), \( u = (u_1, u_2) \), and \( t = (t_1, t_2) \) for the 2-D problem, then substitute in equation (5) to obtain

\[ \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \frac{1}{m^2 + n^2} \begin{bmatrix} k_p m^2 + k_s n^2 & (k_p - k_s) mn \\ (k_p - k_s) mn & k_p n^2 + k_s m^2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \]  
(6)

The above equation, though suited for waves incident at any angle, requires computing the incident angle \( \theta \) for each position vector \( r \), or making a rational approximation to the term \( (\cos \theta)^{-1} \) for each individual Fourier component. In addition, we must discretized the artificial boundary \( \Gamma_a \) using boundary elements, which increases memory space and computing time.

Infinite element technique

The integral equations based on the free space Green’s functions have an ability to consider infinite domain, i.e., the domain which is not bounded by a closed surface. Instead the boundary extends to infinity. Infinite exterior region is considered by using infinite elements where the boundary extends to infinity. Therefore, we need not place the artificial boundary \( \Gamma_a \) at the edges of domain of computation to form a closed surface. This element for the 2-D BE method is shown in Figure 1 where \( r_1 \) is a position vector at the end point of the boundary extending to infinity.

\[ \xi = -1 \quad \xi = 0 \quad \xi = 1 \]

FIG. 1. The infinite element for the 2-D BE method.

The coordinates at a point in the element can be written as

\[ \mathbf{r}(\xi) = N_s^e(\xi) \mathbf{r}_1 + N_r^e(\xi) \mathbf{r}_2, \]  
(7)

where \( N_s^e(\xi) \) and \( N_r^e(\xi) \) are infinite shape functions. The shape functions are given by

\[ \begin{cases} N_s^e(\xi) = -2\xi \frac{\xi}{1 - \xi} & \xi < 0 \\ N_r^e(\xi) = \frac{1 + \xi}{1 - \xi} \end{cases} \]  
(8)

Apparently, from the shape functions we have \( \mathbf{r}(\xi) = \mathbf{r}_1 \) at \( \xi = -1 \), \( \mathbf{r}(\xi) = \mathbf{r}_2 \) at \( \xi = 0 \), and \( \mathbf{r}(\xi) = \infty \) at \( \xi = 1 \). It is noted that the unknown (displacement or stress) in the infinite element varies from node \( r_i \) as follows:

\[ \begin{bmatrix} u \\ t \end{bmatrix} = \begin{bmatrix} f_u(t_1) \\ f_t(t_1) \end{bmatrix}, \]

(9)

where \( f_u \)’s are decay functions which specify the variation of the unknown in the infinite direction from the values \( (u_t, t_1) \) at node \( r_1 \). In general, the displacement and stress for elastic waves are assumed to decay according to

\[ \begin{bmatrix} f_e \to P(r) e^{- \alpha / r} \\ f_t \to P(r) e^{- \alpha / r} \end{bmatrix} \]  
(10)

where, \( P(r) \) is the Lagrangian function. The size of infinite element and the decay parameter \( \alpha \) are determined by calculating frequency.

A quadrilateral infinite element for calculations of the 3-D BI equation or 2-D VI equation is shown in Figure 2 where A and B are the two neighboring nodes at the edge of the curved surface boundary extending to infinity.

FIG. 2. The quadrilateral infinite element.

The position vector at a point in the element can be written as

\[ \mathbf{r}(\xi) = \sum_{i=1}^{N} \gamma_i N_i^e(\xi), \]  
(11)

where \( N_i^e(\xi) \) are 2-D infinite shape functions. The shape functions can be constructed by
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\[ N_1^*(\xi) = \xi_1(1 - \xi_2)/\zeta_1 \]
\[ N_2^*(\xi) = (1 + \xi_1)(1 - \xi_2)/2(1 - \xi_1) \]
\[ N_3^*(\xi) = (1 + \xi_1)(1 + \xi_2)/2(1 - \xi_1) \]
\[ N_4^*(\xi) = \xi_1(1 + \xi_2)/(\zeta_1 - 1) \]

Obviously, the shape functions satisfy \( \sum_{n=1}^{4} N_n^*(\xi) = 1 \), and \( r(\bar{\xi}) = \infty \) at \( \xi_1 = 1 \).

Examples

To show the effectiveness of the above infinite element algorithm for the absorbing boundary, we give two examples calculated by the BE method, including an acoustic model (in Figure 3) and an elastic model (in Figure 4).

Conclusions

An absorbing boundary scheme by using the infinite element algorithm has been given for solutions to the elastic wave integral equations based on the free space Green’s functions. The integral equations have the ability to include infinite domains. We place an infinite element at the end point of the boundary extending to infinity. Then infinite shape functions are constructed for the infinite element. This absorbing boundary scheme do not require the computation of incident angles, the separation of vector fields into compressional and shear components, and the assumption of elastic medium being homogeneous in the region adjacent to the boundary. Numerical experiments have shown that the infinite element-based absorbing boundary is much more absorptive than conventional absorbing boundaries, and takes less internal storage and less computing time.

FIG. 3a. The geometry of a 2-D complex model. The velocity unit is m/s.

FIG. 3b. The acoustic modeling result for CMP records.

FIG. 4a. The geometry of a 2-D model with two layers for elastic wave modeling.

FIG. 4a. The geometry of a 2-D model with two layers for elastic wave modeling.
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References


Sinica, 37, 521-529.


FIG. 4b-4c. The elastic modeling data, where (b) is the $u_x$-component, and (c) the $u_y$-component.