Preface

*Imaging and Inversion: Connection in the local angle domain*

This is the 13th technical report of the “Modeling and Imaging Project” from the Modeling and Imaging Laboratory (MILAB), Center for Study of Imaging and Dynamics of the Earth (CSIDE), Institute of Geophysics and Planetary Physics (IGPP), University of California at Santa Cruz (UCSC).

This year’s technical report is the summary of the final year of WTOPI-Phase III and an introduction to the first year of WTOPI-Phase IV. This year’s results include not only the progress based on our previous work in beamlet propagation/imaging, true-reflection in local angle domain, one-way propagation and modeling, but also work in new directions, especially new theory and methods which serve as seeds and seedlings in our research for WTOPI-Phase IV. The main theme of this year’s report is “Imaging and inversion: connection in the local angle domain”. We continued working in true-amplitude, true-reflection imaging, and made progress in both the acquisition-aperture correction and the study of the true-amplitude propagator. We applied the aperture correction not only to the beamlet migration or GSP (generalized screen propagator) migration, but also to the Gaussian beam migration. The results have been very promising. In addition, in order to take full advantage of working in the local angle domain, we worked out a theory of scattering tomography, which generalizes and extends the classic diffraction tomography in homogeneous background with infinite acquisition-aperture, to the case of arbitrarily heterogeneous media with limited acquisition-aperture. The theory can also converge to the true-amplitude, true-reflection imaging in the local angle domain for special cases after some approximations. Therefore, the theory builds the foundation for future research on true-reflection imaging, parameter inversion and velocity updating. We will continue to study and learn other inversion theories and techniques, such as the inverse scattering theory and waveform inversion developed by other research institutions, to further nourish our new seedlings, and expect a full blooming and harvest in phase-IV.

The volume is mainly composed of three parts:

Part I is devoted to the true amplitude, true reflection imaging (three papers). The first paper is on the application of the aperture correction and improvement in the true-amplitude propagator with the local WKBJ correction. Both corrections are applied to the true-reflection imaging of the SEG/EAGE salt model. Although the aperture correction has the most significant impact on the image fidelity and overall quality, the local WKBJ correction also has some noticeable effect on the image amplitude. The second paper is on the acquisition-aperture correction for the Gaussian beam prestack depth migration. It is proved that the aperture correction scheme in local dip-angle domain can be directly applied to the Gaussian beam migration without extra cost for simple structures. For complex structures, the technique needs further study and tests. The last paper in this part is on a depth migration method based on the full-wave reverse-time calculation and the local one-way propagation. We have done some research in reverse-time migration in the past. Due to the recent interest in turning wave imaging, we looked into the problem from
another point of view and proposed some alternative methods for suppressing the migration artifacts created during the imaging process.

Part II is on **inversion and velocity updating in the local angle domain** (three papers). The first paper is on the theory of scattering tomography in a generally heterogeneous media for both the Born model of volume scattering and the Kirchhoff model of boundary scattering. The theory is for arbitrary configurations in the acquisition system with limited apertures. In the case of a homogeneous background with infinite aperture, the theory goes back to classic diffraction tomography. It also includes the true-reflection imaging for boundary scattering model as a special case, i.e. the case in which both the velocity model and the migration propagator are exactly known. This theory paves the way for further development of efficient imaging/inversion algorithms in WTOPI Phase-IV. The second paper is on the application of the inverse scattering theory to velocity inversion through a novel approach of iterative inverse propagation. The one-dimensional example shows the excellent convergence of methods. We are working on the 2D and 3D formulation and testing and will combine scattering tomography with this iterative procedure to develop a velocity inversion without a prior velocity model. The third paper is on velocity updating using a local image matrix and the phase and amplitude information recorded in the matrix. In addition to the amplitude information, a tau-matrix is introduced to store the angle-dependence phase (travel-time) errors at the imaging point in the local angle domain. Through a dip scan in the local image matrix, the dip-angle can be estimated and the velocity model can be updated by the combination of residual moveout and tomographic inversion.

Part III is on the **development and improvement of one-way propagators** (two papers). The first paper is on the development of a super-wide angle one-way propagator with wavefront reconstruction. The algorithm propagates wave alternatively in two orthogonal directions by a regular one-way method and reconstructs the wavefront iteratively to maintain the accuracy during propagation. The examples of modeling turning waves and salt-flank imaging using turning waves proves the validity and feasibility of the approach. The second paper is a preliminary study on a one-way boundary element method for modeling salt primaries and multiples. This is an attempt to solve the difficulty of modeling primary reflections in strong-contrast media. In this way, the pure multiples of salt bodies can be simulated and possibly eliminated, or even better can be utilized to improve the velocity model and imaging quality.

We have also attached the papers, which have close relevance to the Consortium’s research and published (including those “in press”) during this past consortium year, at the end of this volume for the convenient reference.

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TotalFinaElf, Unocal, Veritas, and WesternGeco. We also welcome the new members, who are currently in the processing of joining the Consortium.

Help and support from our collaborators at the Los Alamos National Laboratory and other industrial partners is highly appreciated. The facility support from the W.M., Keck Foundation is acknowledged.

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PART I

TRUE-REFLECTION MIGRATION/IMAGING
True reflection imaging in heterogeneous media: 
acquisition aperture correction and propagator amplitude correction

Jun Cao and Ru-Shan Wu

Summary

Migration in local angle domain based on one-way wave equations is used to study the influence of acquisition aperture correction and propagator amplitude correction on the image amplitude in heterogeneous media. Localized WKBJ correction based on Local Cosine Basis (LCB) beamlet propagator is applied in post- and pre-stack migration to study the influence of propagator amplitude correction on image amplitude. 2D SEG/EAGE salt model is used for demonstration. Numerical results indicate that localized WKBJ correction have some improvement on the image amplitude though not very dramatic for 2D SEG/EAGE salt data, which has relatively small-offset acquisition. The image after acquisition aperture correction is greatly improved in the entire model. The acquisition aperture correction has larger effect on the image amplitude for migration with limited acquisition aperture in general heterogeneous media.

Introduction

Traditional one-way wave equation based migration can provide a reflector map consistent with the real subsurface structure, but provides unreliable reflection (or scattering) strength (or image amplitude) of the reflectors. True-reflection imaging tries to give not only correct location but also correct image amplitude of the reflectors. This bridges the traditional imaging methods and the direct inversion of medium parameters. The factors influencing the image amplitude include propagator errors (e.g., focusing and defocusing by heterogeneity; geometrical spreading; path absorption and path scattering loss; numerical dispersion and numerical anisotropy), acquisition aperture effect, and imaging condition, etc.

Among these factors, the image amplitude errors caused by one-way wave propagators have been extensively studied recently (e.g. Zhang, et al., 2003, 2005; Zhang, et al., 2004; Cao & Wu, 2005). The original one-way wave equations cannot provide accurate amplitude even at the level of leading order asymptotic WKBJ or ray-theoretical amplitudes (Zhang, et al., 2003). WKBJ amplitude is then introduced into the original one-way wave propagators (e.g. Zhang, et al., 2003, 2005; Wu & Cao, 2005; Cao & Wu, 2005; Luo et al., 2005). Traditionally WKBJ solution is derived by asymptotic approximation in smoothly varying $c(z)$ media, where $c(z)$ is the wave speed at depth $z$ (e.g. Morse and Feshbach, 1953; Aki and Richards, 1980; Clayton and Stolt, 1981; Stolt and Benson, 1986). It has also been obtained by introducing an extra amplitude term based on the transport equation of high-frequency asymptotics to the traditional one-way wave equations (Zhang, 1993; Zhang, et al., 2003). WKBJ solution is also derived from the conservation of energy flux in smooth $c(z)$ media and generalized to general heterogeneous media in local angle domain by introducing the concept of “transparent boundary condition” and “transparent propagator” (Wu & Cao, 2005; Cao & Wu, 2005). In general heterogeneous media, there is no global wavenumber, so beamlet method, which has the localized wavenumber and location information, could be used to implement the localized WKBJ correction. Although the transparent boundary condition does not reflect the physical reality, it may be useful and preferred for some inversion or true-reflection imaging procedure because we can conserve all
the energy collected by the receiver array to the maximum degree. The concept of transparent propagator is similar to the flux-normalized propagator for general heterogeneous media (Wapenaar & Grimbergen, 1996; Wapenaar, 1998). This transparent boundary condition could be the best strategy for imaging and inversion since we do not want to have further loss of energy for the already weak signals during the imaging process (Wu et al., 2004).

With the “true-amplitude” one-way wave equations, better image amplitude is obtained in common-shot migration (Zhang, et al., 2003, 2005; Cao & Wu, 2005) and common-angle gathers (Zhang, et al., 2004). Most true-amplitude propagators are formulated and implemented in space domain. The localized WKBJ correction in general heterogeneous media proposed by Wu & Cao (2005) is in local angle (wavenumber) domain. Numerical results demonstrated that WKBJ correction can improve the image amplitude in smooth \( c(z) \) media though not as much as the acquisition aperture correction does (Cao and Wu, 2005). The calculated impulse responses in general heterogeneous media demonstrate the effect of localized WKBJ correction on wavefield amplitude (Luo et al., 2005).

The limited data acquisition aperture in reality also influences the amplitude of the image. Wu et al. (2004) proposed an amplitude correction method in local angle domain by acquisition aperture correction. The numerical examples showed significant improvement in both the total strength images and the angle-dependent reflection amplitudes, which demonstrated the significance of aperture correction in true-reflection imaging.

In this paper, the influence of acquisition aperture correction and propagator amplitude correction on the image amplitude in heterogeneous media is studied. We begin with a brief review of the theory of localized WKBJ correction based on LCB beamlet propagator and the theory of acquisition aperture correction in local angle domain. Then we apply the beamlet propagator with localized WKBJ correction to post- and pre-stack migration for 2D SEG/EAGE salt model to study its influence on image amplitude. Finally we compare the influences of localized WKBJ correction and acquisition aperture correction on the image amplitude.

**Localized WKBJ Correction Based on Beamlet Propagator**

The original WKBJ correction in smoothly varying \( c(z) \) media can be written as

\[
\frac{P_z}{P_0} = \sqrt{\frac{\cos \theta_1 \rho_2 c_z}{\cos \theta_2 \rho_1 c_1}} = \sqrt{\frac{k_z(c_z)}{k_1(c_1)}} , \tag{1}
\]

where \( P, \rho, c, \theta, k_z \) are pressure, density, velocity, propagation angle and global vertical wavenumber, respectively (see Figure 1).

Equation (1) can be generalized to general heterogeneous \( c(x, z) \) media based on beamlet propagator in local angle domain (Wu & Cao, 2005; Cao & Wu, 2005; Luo et al., 2005). The total wavefield can be decomposed into beamlets (e.g. Wu et al., 2000)

\[
u(x, z, \omega) = \sum_n \sum_m u(\bar{x}_n, \bar{\omega}_m, z, \omega) b_{mn}(\bar{x}_n, \bar{\omega}_m), \tag{2}
\]

where \( b_{mn} \) is the decomposition basis vector (beamlet), \( u(\bar{x}_n, \bar{\omega}_m) \) is the coefficient of the decomposed beamlet located at space \( \bar{x}_n \) and wavenumber \( \bar{\omega}_m \), where
\[ \bar{x}_m = n\Delta_x, \quad \bar{\xi}_m = m\Delta_z. \] (3)

The localized WKBJ correction based on beamlet can be written as

\[ \frac{u(\bar{x}_m, \bar{\xi}_m, z + \Delta z, \omega)}{u(\bar{x}_m, \bar{\xi}_m, z, \omega)} = \left( \frac{\rho(\bar{x}_m, z + \Delta z)}{\rho(\bar{x}_m, z)} \right) \frac{k_z(\bar{x}_m, z)}{k_z(\bar{x}_m, z + \Delta z)}, \] (4)

where \( k_z(\bar{x}_m, z) \) is the window location \( \bar{x}_m \) dependent vertical wavenumber at depth \( z \), which satisfies

\[ k_z^2(\bar{x}_m, z) = \omega^2 / v^2(\bar{x}_m, z) - \bar{\xi}_m^2. \] (5)

Figure 1: Diagram for WKBJ correction.

**Acquisition Aperture Correction in Local Angle Domain**

The imaging condition (for a single frequency) in local angle domain (Wu & Chen, 2002, 2006) can be written as,

\[ L(\bar{x}, \bar{\theta}, \bar{\theta}_s) = 2 \sum_{x_i} G_i^*(\bar{x}, \bar{\theta}_s; x_i) \cdot \int_{A(x_i)} dx_g \frac{\partial G_i^*(\bar{x}, \bar{\theta}_s; x_g)}{\partial z} u_s(x_g; x_i), \] (6)

where \( G_i \) is Green’s function used in the imaging process, which could be different from the Green’s function of forward modeling; \( \bar{\theta}_s \) and \( \bar{\theta}_r \) are the source and receiving angles, respectively; “*” stands for complex conjugate; \( G_i(\bar{x}, \bar{\theta}_s; x_i) \) is the incident wavefield in the local angle domain at the imaging point \( \bar{x} \); and the integral is a back propagation Rayleigh integral, \( A(x_g) \) is the spatial receiver aperture and \( u_s(x_g; x_i) \) is the recorded scattered waves at receiver \( x_g \) from the source at \( x_i \) on the surface. The relevant amplitude correction factor matrix \( F_a \) for imaging condition (6) is,

\[ \left| F_a(\bar{x}, \bar{\theta}, \bar{\theta}_s) \right| = 2 \sum_{x_i} \left| G_i^*(\bar{x}, \bar{\theta}_s; x_i) G_{F}(\bar{x}, \bar{\theta}_s; x_i) \right| \left( \int_{A(x_i)} dx_g \left| G_F(\bar{x}, \bar{\theta}_s; x_g) \right|^2 \right)^{1/2}, \] (7)

where \( G_F \) is Green’s function for forward modeling. Since the superior performance of the dip-angle domain correction scheme (Wu & Luo, 2005), we will apply the acquisition aperture correction in dip-angle domain here.

**Imaging with Propagator Amplitude Correction and Acquisition Aperture Correction**
In this part, we apply the LCB beamlet propagator with localized WKBJ correction to post- and pre-stack migration for 2D SEG/EAGE salt model (Figure 2) to study the influence of localized WKBJ correction on image amplitude. Then we compare its influence with that of acquisition aperture correction on image amplitude.

(1) Propagator amplitude correction in post-stack migration

Post-stack migration is a wave back-propagation process, so it can directly show the effect of localized WKBJ correction on the amplitude of the propagator. Figure 3 shows the post-stack migration result before and after localized WKBJ correction. The image amplitude is obviously improved in the entire model, especially that of the steep faults in the sediment, the salt boundary and the subsalt structures. WKBJ solution is a transparent propagator (or energy-conservative propagator), which neglects all the scattering (reflection) loss during propagation so we can conserve all the energy collected by the receiver array to the maximum degree. The side-effect of WKBJ correction is that the artifact is also amplified.

Figure 2: 2D SEG/EAGE salt model

Figure 3: Post-stack image for 2D SEG/EAGE salt model before and after localized WKBJ correction.
Propagator amplitude correction and aperture correction in pre-stack migration

Figure 4 shows the pre-stack migration result with convolution imaging condition before and after localized WKBJ correction. Similar to post-stack case, the image amplitude is improved in the entire model and also with stronger artifact. To make the image amplitude closely represent the reflection strength, the image is normalized with the square of incident wave amplitude in local dip angle domain. The normalized images with/without the localized WKBJ correction (Figure 5) give very similar amplitudes. The smaller-offset acquisition for the SEG/EAGE salt model data (maximum offset 14000 feet) may be the reason of the relatively weak effect of WKBJ correction on image amplitude. The small-offset acquisition can only acquire reflected waves with small reflection angles from the dipping reflectors, which makes the paths of incident wave and reflected wave be quite similar. And the lateral velocity variation is not very dramatic either. Therefore the effect of WKBJ correction for the incident and reflected wave is similar, which makes the image amplitude correction less dramatic. However, we notice the image amplitude increase for the subsalt steep faults. This will be seen clearer later in conjunction with the aperture correction.

Finally, localized WKBJ correction and acquisition aperture correction are both applied to pre-stack migration. The result with both corrections (Figure 6b) shows similar image to that only with acquisition aperture correction (Figure 6a). WKBJ correction also amplifies the artifact in subsalt area. Compared with the image before acquisition aperture correction (Figure 4a), the image after acquisition aperture correction (Figure 6a) is greatly improved in the entire model. The image for the steep faults in the sediment is sharper and more continuous. For subsalt structures, the image along the steep sand structures and the baseline are much more uniformly distributed after the correction. And the noises in the subsalt region caused by salt body multiples are also reduced. This shows that the acquisition aperture correction has larger effect on the image amplitude for 2D SEG/EAGE salt model. This agrees with the result in smooth $c(z)$ media (Cao & Wu, 2005). One noticeable improvement in image amplitudes is the two steep subsalt faults directly beneath the salt body. The phase accuracy of the waves is improved so that the reflected signals from those reflectors can be better focused after back-propagating through the irregular salt body. The WKBJ correction helps in restoring the strength of large-angle signals. As a result, the image amplitudes of these steep faults are increased, although the noise background is also raised.

Conclusion

We apply localized WKBJ approximation based on LCB beamlet propagator in post- and pre-stack migration for 2D SEG/EAGE salt model to study its influence on image amplitude. The results indicate that localized WKBJ correction have some though not very dramatic improvement on the image amplitude for 2D SEG/EAGE salt model. The smaller-offset acquisition and not very dramatic lateral velocity variation for the SEG/EAGE salt model data may be the reason of the relatively weak effect of WKBJ correction on image amplitude. The side-effect of WKBJ correction is the artifact is also amplified. With acquisition aperture correction the image is greatly improved in the entire model. The acquisition aperture correction has larger effect on the image amplitude for migration with limited acquisition aperture in general heterogeneous media. The combined amplitude corrections of localized WKBJ and acquisition aperture can give the best result, especially for the weak steep reflectors directly.
beneath the salt body in the case of SEG/EAGE salt model.

Figure 4: Pre-stack image for 2D SEG/EAGE salt model before and after localized WKBJ correction.

Figure 5: Pre-stack image for 2D SEG/EAGE salt model before and after localized WKBJ correction. The image is normalized with the square of incident wave amplitude in local dip angle domain.
Figure 6: Pre-stack image with acquisition aperture correction for 2D SEG/EAGE salt model before and after localized WKBJ correction.

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References


Prestack Gaussian beam migration and amplitude compensation in local angle domain for acquisition aperture correction

Yingcai Zheng and Ru-Shan Wu

0. Abstract

Prestack Gaussian beam (GB) migration has been investigated in local angle domain for a 2D medium. The ray property of the GB equips itself with propagation angle information. We can directly form the local image matrix (LIM), through which the acquisition aperture correction to the image amplitude is performed. The image amplitude after aperture correction is in good agreement with the velocity contrast at an interface. The aperture correction by GB requires no additional computation cost. Such correction process can be formed during the imaging process in the local angle domain.

1. Introduction

A Gaussian beam is an asymptotic solution to the parabolic wave equation (Tartarskii, 1971) constructed in a curvilinear coordinate system (i.e., ray-centered coordinates) dictated by the dynamic ray-tracing differential equations. It has been widely known that this method overcomes some critical difficulties raised in the traditional Kirchhoff migration scheme, in particular, the two-point boundary value problem and the caustic problem (e.g., Popov, 2002; Cerveny et. al., 1982). The GB approach has been used and validated in both forward modeling (e.g., Nowack, 1985) and its reverse process, migration (e.g., Hill, 1990; Hill, 2001; Gray, 2004; Nowack, 2006). Despite of its good nature, which combines both ray and wave properties, the beam solution sometimes can be inaccurate if large velocity fluctuations exist within the beam width. This can be caused either by very complex geological structures or simply due to the fact that the beam width is diverging with propagation distance. To solve this difficulty, Han and Wu (2005) proposed a wavefield reconstruction procedure using GBs based on the idea of survey sinking (Claerbout, 1984), which comprises a two-stage process, a downward continuation of recorded wavefields and a downward continuation of the shots using the reciprocity theorem. The wavefield continuation is implemented by GB propagators. After the sinking process, we can then decompose the wavefields again into GBs and propagate them downwards. If we carry out the reconstruction procedure at every depth level, this will be analogous to the beamlet migration. However, because the GB has both local and global properties, we can choose any reconstruction depths.

The local angle information of the migrated image is important in investigation of the medium properties and velocity building and updating. The ray property of the GB naturally equips itself with local angle information and the ability to handle turning waves. The wave property can achieve better accuracy in wavefield extrapolation. Using GBs, we can directly obtain the image matrix in local angle domain, without further performing a local plane wave analysis such as in one-way wave equation methodology (e.g., Wu and Chen, 2006; Xie et. al., 2005). The inaccuracy of the image amplitude comes from two sources, the incorrect propagators and limited data acquisition aperture. It has been demonstrated that the latter effect is more important
in image amplitude compensation analysis (Wu et. al., 2004; Cao and Wu, 2005). In this paper, we emphasize local angle domain image formed by GBs in a two-dimensional model and the compensated image amplitude after aperture correction (Wu et. al., 2004).

2. Prestack Gaussian beam migration in local angle domain

Before we proceed into any further discussion, it is useful to briefly review some basic characters of GB solution and its construction procedure. We first trace a ray by solving the canonical Hamiltonian differential system,

\[ \dot{x}(s) = v(s)\dot{p}(s), \]  

\[ \dot{p}(s) = -\frac{1}{v^2(s)} \nabla V(\dot{x}). \]  

The “dot” is the Newtonian notation for differentiation with respect to ray path length \( s \). \( \nabla \) is the Laplacian for spatial gradient. \( G_x \) and \( G_p \) are Cartesian position and slowness vector at \( s \), respectively. We use \( \tau(s) \) to denote traveltime. Ray curvature and beam width are sought by the following equations,

\[ Q \cdot \dot{s}(s) = v(s)P(s), \]  

\[ P \cdot \dot{s}(s) = -\frac{1}{v^2(s)}Q(s)\frac{\partial^2 V(s,n)}{\partial n^2}. \]  

\( n \) is another ray-centered coordinate, which measures the perpendicular distance for a receiver to a ray. The final GB solution reads,

\[ u(s,n,\omega) = A(s)\exp\left[i\omega\tau(s) + \frac{i\omega P(s)}{2Q(s)}n^2\right]A^*(s) = \sqrt{\frac{v(s)}{Q(s)}}. \]  

The imaginary part of the ratio \( P/Q \) controls the beam width and its real part measures the traveltime induced by local curvature of the wavefront.

In migration, we decompose the recorded wavefield for a given shot into GBs using local slant stack technique (e.g., Hill, 1990; Hale, 1992) and then propagate them individually. Such decomposition can also be accomplished by frame-based theory (e.g., Lugara et. al., 2003; Nowack et. al., 2006). The final wavefield at a receiver is the summed contribution from all the beams. Likewise, we apply GB representation to wavefield radiated from a point source (see Cerveney, 1982). The crosscorrelationes by beams from the source (\( s.b. \)) and those from receivers (\( r.b. \)) form a prestack image matrix,

\[ I(\xi;\theta_s;\theta_g) = \sum_{s.b.} \sum_{r.b.} u^*(\theta_s)u^g(\theta_g) = \sum_{s.b.} \sum_{r.b.} A^*(\theta_s)A^g(\theta_g)e^{i\omega[\phi^s(\theta_s) + \phi^g(\theta_g)]}, \]  

where we use superscripts or subscripts \( s \) and \( g \) to designate the beam from source or from receivers; \( A^s \) and \( A^g \) are used to denote complex amplitude \( A \) in equation (3); \( \phi^s \) and \( \phi^g \) are for the complex phase term in equation (3) and \( \theta_s, \theta_g \) the local angles of \( s.b. \) and \( r.b. \) at the image point \( \xi \). These two angles are defined with respect to the vertical axis (see figure 1).
3. Aperture Correction in Local Dip Angle Domain

Once the local image matrix $I(\xi; \theta_s; \theta_g)$ is available, we can correct the image amplitude impaired by the finite acquisition aperture. The local scattering matrix (LSM) $S$, which measures the intrinsic medium properties, can be related to the LIM (see Wu et. al., 2004),

$$I(\xi; \theta_s; \theta_g) = S(\xi; \theta_s; \theta_g) |u'(\xi; \theta_s)|^2 |B(\xi; \theta_g)|,$$

and

$$|B(\xi; \theta_g)|^2 = \int_{x_g} dx_g |u^2(\xi; \theta_g)|^2.$$  \hspace{2cm} (6)

In the following numerical example, we correct the image amplitude in local dip angle domain. The local dip angle $\theta_n = (\theta_s + \theta_g)/2$ is the angle between the vertical axis and the vector that bisects the incident angle $\theta_s$ and the receiving angle $\theta_g$. The reflection angle is $\theta_r = |\theta_s - \theta_g|/2$.

The image after aperture correction reads,

$$|S(\xi)|^2 = \sum_{\theta_n} \frac{1}{D(\xi; \theta_n)^2} + \varepsilon^2 \sum_{\theta_n} |I(\xi; \theta_s; \theta_g)|^2,$$

where,

$$|D(\xi; \theta_n)|^2 = \sum_{\theta_n} |u'(\xi; \theta_s)|^2 |B(\xi; \theta_g)|^2.$$ \hspace{0.75cm} (8)

In the above formulation, there is an implicit summation sign over all the sources for angle domain image $I(\xi; \theta_s; \theta_g)$ and the correction factor $D(\xi; \theta_n)$. Notice that the correction factor $D(\xi; \theta_n)$ can be directly formed along with the imaging process in angle domain. Therefore, no additional cost is needed to compute this factor.

4. Numerical Example for a Four-Layer Model

We use a 4-layer model (see figure 2; same interface geometry as in Baina et. al., 2002) to test our algorithm. These four interfaces separate 5 homogeneous regions and the velocities read from shallow to deep, 3500, 3700, 4000, 4200 and 4500 m/s, respectively. A synthetic dataset is calculated using the full finite difference method. There are 201 shots whose starting position is at x=0 km, followed by subsequent shots every 50 meters. 481 Receivers are deployed symmetrically about the source position. The spatial distance between adjacent receivers is 25
meters. The main frequency of the Ricker source wavelet is 17.5 Hz. We use symbol T1, T2, T3 to denote the top three non-flat interfaces. We produce a prestack migration image and pick out the image amplitudes along these three interfaces (figure 3). The amplitude varies dramatically, though the velocity contrast along each interface is uniform. However, after the aperture correction, the image amplitudes along the interfaces become flat. A significant improvement has been done for the image of rugged T1 interface (figure 4). The image amplitude around the valley and the peak are uniform after correction.

5. Conclusions

The ray property of the GB is naturally associated with the angle-domain image and the ability to handle turning waves. Its wave property makes possible an accurate seismic wavefield extrapolation. The local and global properties of the GB, together with survey sinking techniques could give us better control on the beam width. The aperture correction factor can be obtained with no additional computation cost, during the angle domain imaging process. Numerical example based on Gaussian beam migration presented here shows a significant improvement on the image amplitude.
Figure 3. Picked image amplitude after aperture correction (top) and before aperture correction (bottom).
Figure 4. Comparison between image after aperture correction (top) and image with no aperture correction applied (bottom).

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A depth migration method based on the full-wave reverse-time calculation and local one-way propagation

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Summary

A depth migration method combining the full-wave reverse-time calculation and local one-way propagation is proposed. The full-wave finite-difference propagator is used to extrapolate the source and receiver waves in the model. The wavefields are extracted from selected locations and then a local one-way propagator is used to extrapolate the wavefield towards different directions. The one-way propagator is also served as a one-way filter, separating waves propagating along different directions. Finally, the image is calculated from directional waves generated by the one-way propagator and the artifacts from backscattered waves of both source and receiver sides are effectively eliminated. This method also dramatically reduces the size of output files which is important when dealing with large 3D models.

Introduction

The full-wave reverse-time migration method does not have the angle limitation of one-way propagators and can be used to image structures in complex areas especially for steeply dipping structures where turning waves are involved. Although the reverse-time migration is a relatively expensive method, the development of computer hardware and software (remarkably the popularities of PC clusters and Linux operating system) plus the ever increasing interests to explore complex structures in challenging regions stimulated recent interest to this method (e.g., Wu, et al., 1996; Mufti, et al., 1996; Sun and McMechanz, 2001; Biondi and Shan, 2002; Yoon, et al., 2003, 2004; Mulder and Plessix, 2004; Fletcher, et al., 2005; Wang, et al., 2005; Hou and Marfurt, 2005).

One of the major problems that affect the application of reverse-time migration is that the wide-angle capability of full-wave propagator can generate spurious cross correlations from diving waves and back-scattered waves, and causes serious artifacts (Yoon, et al., 2004; Mulder and Plessix, 2004; Fletcher, et al., 2005; Wang, et al., 2005). Special techniques such as velocity model smoothing or interface impedance matching have been introduced to solve this problem. These approaches can improve the situation in poststack migration where only one wavefield is involved, but the result is unsatisfied when applied to the prestack migration where cross correlation happens between the source and receiver waves.

Recently, several approaches have been proposed to solve this problem. Mulder and Plessix (2004) suggested that these artifacts can be removed by blanking the data, high-pass filtering or iterative migration. Yoon, et al. (2004) proposed to use an angle related weighting function calculated from Poynting vectors of source and receiver waves to pick the right events for imaging. Fletcher, et al. (2005) tried to eliminate the artifacts by introducing a directional damping term in areas where unwanted reflections occur. Wang et al. (2005) proposed a pseudo-space method by using a space distortion to transform a heterogeneous velocity model into a homogeneous model where no reflections were generated.
In this paper, we propose to use the local one-way propagator as the filter to separate the full-wave equation generated wavefield into directional waves. Then use directional wavefields for imaging. By making proper combinations of these directional waves, the artifacts caused by undesired waves can be avoided. Another advantage of this method is that it only needs to store part of the wavefield. The size of the file containing the wavefield can be reduced by about an order-of-magnitude which dramatically reduces the effort in input/output, network traffic and data management.

**Methodology**

The wavefield in a closed region without internal source can be calculated from its boundary values using the Rayleigh integral

\[
    u(r) = \int_S \left( u(\xi) \frac{\partial G(r,\xi)}{\partial n} - G(r,\xi) \frac{\partial u(\xi)}{\partial n} \right) ds,
\]

where \( u(r) \) is the wavefield, \( G(r,\xi) \) is the Green function, \( S \) is the boundary, \( r \) is the position of an internal point, \( \xi \) is the position on the boundary, and \( \partial \)/\( \partial n \) is the derivative along the boundary normal. From the wavefield calculated using the full-wave propagator, choose the wavefield along certain boundaries, the wavefield in the entire region can be recovered from these boundary values. If we choose the boundaries along different directions and use one-way propagator as the Green function in equation (1), we can obtain waves propagating along different directions. For convenience, we use the wavefield on horizontal lines (or planes for 3D case) for wavefield extrapolation in near vertical direction and use wavefield on vertical lines for near horizontal extrapolation. By choosing different normal directions of the boundary and using forward or backword one-way propagators, we can extrapolate wavefield in forward or backword directions.

![Figure 1. A simple two-layer velocity model. The horizontal and vertical lines indicate the locations where wavefields are needed for one-way extrapolations.](image)

Taking the horizontal line as an example, the wavefield \( u(\xi) \) and \( \partial u(\xi)/\partial z \) are obtained from the full-wave calculation. Substituting these values into equation (1) and using one-way Green function \( G^{+z}(r,\xi) \) and \( \partial G^{+z}(r,\xi)/\partial z \), where superscript \( +z \) denotes the one-way propagator along positive \( z \) direction, we can calculate wavefield below that horizontal line. Similarly, using Green function \( G^{-z}(r,\xi) \) and \( \partial G^{-z}(r,\xi)/\partial z \), we can calculate wavefield above the horizontal line and along
Using wavefield along horizontal lines at different depth levels, we can reconstruct wavefields along $+z$ and $-z$ directions between these lines. A similar method can be used to process the near horizontally propagating waves. In these cases, the one-way propagator only deals with narrow-angle propagation and the wavefield is extrapolated only for a very short distance. The accuracy of the one-way propagator will not seriously affect the result.

![Figure 2. Wavefield snapshots calculated using full-wave finite-difference method. (a) and (b) are source wavefields at 0.8 s and 1.2 s. (c) and (d) are receiver wavefields at $T-0.8s$ and $T-1.2s$.](image)

**Numerical Examples**

To show how this system works, we first conduct the migration for a simple two-layer model. Figure 1 shows the velocity model. The size of the model is $451 \times 150$ with $dx = dz = 24$ m, and the velocities are 3.0 km/s and 4.0 km/s for upper and lower layers, respectively. The horizontal and vertical lines in the velocity model indicate the location where the wavefield is picked. Both the synthetic data and the reverse-time migration are calculated using a fourth-order full-wave finite-difference code. To focus our attention to the effect of back-scattered waves, We mute the first
arrival from the shot record. Shown in Figure 2 is migrated wavefield using the full-wave method, with 2a and 2b are source waves at 0.8 s and 1.2 s, and 2c and 2d are receiver waves at $T-0.8s$ and $T-1.2s$, where $T$ is the maximum recording time. We then output the wavefield on horizontal lines at depths for every 15 $dz$ and use a one-way propagator (Xie and Wu, 1998, 2005) to extrapolate the wavefield from each depth to the next depth (15 steps). Figures 3a and 3b are the reconstructed wavefield $u_s^z(r)$, and 3c and 3d are the reconstructed wavefield $u_g^z(r)$, where subscripts s and g denote source and receiver waves. Compared with Figure 2, the undesired reflections have been eliminated.

The image can be calculated from directional wavefields. To avoid artifacts from cross correlations between diving waves and back-scattered waves, we choose wave pairs related to the true reflections, for example $u_s^z(r)u_g^z(r)$, $u_s^z(r)u_g^{-z}(r)$, $u_s^z(r)u_g^+(r)$ or $u_s^{-z}(r)u_g^+(r)$, to form the image. The final image can be obtained by summing up all partial images. Shown in Figure 4 are reverse-time images for the two-layer model, where 4a is calculated using the conventional zero-lag cross correlation image condition

$$I(r) = \int_0^T u_s(r,t)u_g(r,T-t)dt,$$

where $I(r)$ is the image, $u_s$ and $u_g$ are source and receiver waves. Figure 4b is calculated using the image condition given by Yoon et al. (2004). In Figure 4c, the image is calculated from directional waves $u_s^z(r,t)u_g^{-z}(r,T-t)$ reconstructed using local one-way propagator. In Figure 4a, there are artifacts shown above the interface, while in 4b and 4c, they have been properly removed.

As second example, we calculate the image of a simple salt dome model. Figure 5 shows the velocity model overlapped with the wavefield snapshot from the finite-difference forward modeling. The velocity model composed of a background with a vertical velocity gradient and a salt dome with steep flanks. The size of the model is the same as the two-layer model and major phases from different part of the structure are indicated with rays. Figure 6 shows the wavefield from finite-difference forward modeling. In the successive time slices, we can see the large angle reflections developed from the salt flank. Even for such a simple velocity model, the reflections are rather complex. To properly image the steep flank of the salt dome, wide-angle turning waves must be properly handled.

Figure 7 shows the images from depth migrations overlapped with the velocity model. A single source is used to illuminate the model. Figure 7a is calculated using the conventional zero-lag cross correlation imaging condition. Figure 7b is calculated using the imaging condition of Yoon et al. (2004). Figure 7c is calculated from the waves $u_s^z(r)$ and $u_g^z(r)$ reconstructed using the local one-way propagator, and 7d is calculated using $u_s^z(r)$ and $u_g^{-z}(r)$. Note that horizontal structure is better imaged in 7c and vertical structure is better imaged in 7d. The artifacts shown in 7a have been removed in 7b to 7d.
Figure 3. Wavefield snapshots reconstructed using the one-way propagator. (a) and (b) are wavefields along $+z$ and at 0.8 s and 1.2 s. (c) and (d) are wavefields along $-z$ and at $T - 0.8s$ and $T - 1.2s$. Note that compared with the finite-difference result shown in Figure 2, the undesired reflections have been eliminated.
Figure 4. Reverse-time images for the two-layer model, (a) using conventional zero-lag image condition, (b) using the image condition given by Yoon et al. (2004), and (c) using full-wave propagator plus local one-way propagator.

Figure 5. A simple salt dome model with steep flanks. Overlapped is the snapshot generated by finite-difference forward modeling. The major reflection events are marked with rays.
Discussions and Conclusion

We proposed a depth migration method based on the full-wave reverse-time calculation and local one-way propagation. Both the source and receiver waves are extrapolated using the full-wave finite-difference propagator. There is no angle limitation for these wavefields. Then the local one-way propagator is used to separate the waves into different directions. Finally, the image is calculated using directional wavefields. Using the one-way propagator at the imaging stage has the following advantages. (i) It eliminates undesired waves and removes the artifacts. (ii) The input/output file size can be substantially reduced (about an order-of-magnitude smaller than if the entire wavefield is dumped out). The additional computation for the one-way propagation can be compensated by the time saved from the I/O and network traffic. The reduced file size makes the management of the data easier and provides the possibility for repeated later studies.

![Figure 6. Snapshot of finite-difference forward modeling. Note the complex large angle reflections from the salt flank. The downward arrival at the later time is the artifacts from the upper boundary.](image)

The result presented in this paper is preliminary. Further investigation is needed to better reconstruct the wavefield in complex regions using one-way propagator. More sophisticated method such as the angle domain image analysis (Wu, et al. 2004, and Xie, et al., 2005) may be required to combine images from multi-directional wavefields.
Figure 7. Depth migration image for a simple sand dome model, (a) using the conventional zero-lag imaging condition, (b) using the image condition of Yoon et al. (2004), (c) using $u_\theta^x(r)u_\phi^x(r)$ and (d) using $u_\phi^x(r)u_\phi^x(r)$.

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PART II

INVERSION AND VELOCITY UPDATING
Scattering tomography for heterogeneous media in the case of finite data apertures: I. theory

Ru-Shan Wu

Abstract

Scattering tomography for both the Born model of volume scattering and the Kirchhoff model of boundary scattering are formulated. The Kirchhoff model is based on the tangent-plane approximation and the Kirchhoff integral representation. The former model is good for weak volume perturbations and the latter model is suitable for sharp boundaries imbedded in smoothly varying backgrounds. Tomographic inversions derived in this paper can be summarized as a process of backpropagation plus filtering in the local angle domain. The backpropagation is a doubly focusing process, similar to the imaging principle in migration/imaging. The filtering is a deconvolution in the local angle domain, and is different for the Born model of volume scattering and the Kirchhoff model of boundary scattering. The derived decon-filtering formulation is for arbitrarily heterogeneous media, including those with sharp boundaries, with limited acquisition aperture. For the Born model in a homogeneous background with infinite acquisition aperture, the formulation reduces to the classic diffraction tomography. In the case of boundary scattering model, if the velocity model is accurate and the propagators do not have phase errors, the decon-filtering will be similar to the amplitude correction in the true-reflection imaging. It is shown that under weak scattering approximation, the true-reflection images for the boundary scattering model can be formulated into a direct inversion of local velocity contrast along the boundaries.

1. Introduction

Inversion theory and methods are forward-model dependent. Different forward models have different parameters and may end up with quite different inversion schemes. In this paper we will derive the formulation of scattering tomography in heterogeneous media for both volume scattering and boundary scattering. The formulation includes the case of finite data aperture (finite frequency-band and finite spatial aperture of acquisition system). For the case of volume scattering, we will show the relation of the new formulation with the classic diffraction tomography. For the case of boundary scattering, the scattering tomography is closely related to the true-reflection imaging with amplitude correction in local angle domain.

2. Born Modeling (Volume Scattering Model)

The parameters to be inverted are volume perturbations of unknowns with respect to a given reference medium. The Green’s function is the impulse response in the reference medium. In the Born model, the scatterings of each volume elements are independent from each other and no interaction between the elements is taken into consideration. The elastic scattered field $U(x, x)$ (displacement vector) under the Born approximation, measured on the surface at $x$, excited by the source at $x$, can be written as
The far-field P-P scattering with a unit plane wave incidence for an isotropic elastic medium is given as (see e.g. Wu and Aki, 1985; Wu, 1989)

\[
U^{\text{pp}}(x_g, x_s) = \frac{k_0^2}{4\pi} g^a \int dV(x') \hat{\mathbf{i}} \cdot \nabla \delta \rho(x') G(x_g, x') \cdot G(x_s, x')
\]

In these equations, the volume integration is over the whole volume of the heterogeneity, \( G \) is the elastic Green’s function, \( \nabla \cdot G \) is the divergence of \( G \), \( \nabla G \) is gradient of \( G \), and \( \nabla G \) is the transpose of \( \nabla G \), \( \hat{\mathbf{i}} \) is unit vector along the incident direction, \( \hat{\mathbf{o}} \), unit vector in the scattering direction, \( \rho_0 \) is the background density, \( \lambda_0 \) and \( \mu_0 \) are the elastic Lame constants of the background medium, \( \delta \rho \), \( \delta \lambda \) and \( \delta \mu \) are the corresponding perturbations, \( k_0 = \omega / \alpha_0 \), and \( g^a \) is the scalar Green’s function

\[
g^a(x, x^0) = \frac{1}{r_0} \exp[i\omega r_0 / \alpha]
\]

with \( r_0 = |x - x^0| \) and \( x^0 \) is center of the heterogeneity (inclusion) under consideration. Elastic Born scattering (1) can be expressed in the form of other parameter perturbations, such as the density, P-wave and S-wave velocities (Sato and Fehler, 1998) or density, P- and S-wave impedances (Tarantola, 1987, 2005).
3. Kirchhoff Modeling (Boundary Scattering Model)

Many kinds of heterogeneous media in the earth can be decomposed into a rough part and a smooth part. The rough part is composed of irregular sharp interfaces. The smooth part is formed by many smoothly varying domains surrounding by those sharp interfaces. The smoothly varying parts of the medium cause diffraction, path bending, travel-time variation, amplitude changes of the wavefield (forward scattering), but only sharp boundaries produce significant backscattering (reflection) and large-angle scattering. Therefore the roles of forward scattering and backscattering in this model are very different. *Forward scattering only modifies the Green’s function of the reference medium, while backscattering of the boundaries is actually responsible for the generation of reflection signals.* In contrast, in volume scattering model the reflection signals are formed by the interference (coherent superposition) of volume scattered fields.

**Elastic wave boundary scattering**

Although multicomponent data are available in the industry, today most data processing is still dealt with the pressure data.

Even with the pressure data, we still need to set up the inversion parameters as elastic ones from the very beginning. Since the scattering behavior, such as the angle and frequency dependences are radically different for acoustic and elastic media.

We will first formulate and test the P-wave scattering problem and then move to the full elastic wave scattering. In order to compare with some theory on acoustic wave inversion, and to apply the theory to some synthetic data of acoustic model, we also consider the case of scalar wave scattering.

**Kirchhoff representation integral**

Start from the surface representation integral, the Kirchhoff integral, for the pressure field:

\[
p^{sc}(x) = p(x) - p^{0}(x) = \int_{\Gamma} p^{sc}(x') \frac{\partial}{\partial n} G(x, x') ds(x') - \int_{\Gamma} G(x, x') \frac{\partial}{\partial n} p^{sc}(x') ds(x'),
\]

where \( p \) is the total field, \( p^{0} \) is the incident field, and \( p^{sc} \) is the scattered field, \( x \) and \( x' \) are the position vectors of the observation point and scattering point on the boundary respectively, \( ds(x') \) is a boundary element at \( x' \) on the surface \( \Gamma \), \( G(x, x') \) is the Green’s function for the background region (smooth region), \( \partial/\partial n \) denotes differentiation with respect to the outward normal of the interface at \( x' \). Another form of the Kirchhoff integral is to use the total field instead the scattered field inside the integral. However, in the cases of our interest, the observation point is in the same side of the integration surface as the source, and therefore the contribution of the integration on incident field is null. In the following derivation, we will always use the form expressed in equation (6). The Kirchhoff integral gives the solution at any point above the curved interface if we know the scattered field \( p^{sc} \) and its normal gradient.
\( \partial p^\text{ac} / \partial n \) at all the points along the surface. In mathematical rigorosity, the surface integral has to be along a closed surface. In practice, if the surface is extended laterally far enough and no end point scattering is observed in the measurement, we can close the surface with a large semicircle having nearly infinite radius.

In order to model the scattered field, we need to calculate the total pressure field and its normal gradient from the incident field at all the points on the surfaces. Rigorously, we should solve the integral equation along all the interfaces. This will take into account of the interactions between all the boundary elements. However, the resulted equation is usually a huge matrix equation and computationally intensive if not tractable. Tangent plane approximation (Kirchhoff approximation) provides us a convenient tool for efficient modeling and inversion.

**Tangent plane approximation**

Boundary scattering can be formulated in different ways. Kirchhoff model is a convenient approximation from the viewpoint of inversion theory. Kirchhoff theory is known as *physical optics approximation* or *tangent plane approximation* (Ogilvy, 1991). **Tangent plane approximation** states explicitly its meaning. At any point of a scattering boundary (interface), the local reflection is treated as by an infinite plane tangent to the local surface. In this sense the theory is a high-frequency asymptotic solution of a smooth interface approximation. Therefore the theory is accurate for the smoothly varying interface but has limited use for rough surfaces. In scattering tomography, we will use this approximation to relate the migrated images in local angle domain to the local interface property (elastic parameters). In the meanwhile the velocity structure of the overlaying structure is also updated by updating the propagators (Green’s functions).

**Tangent plane approximation for plane wave scattering**

For a plane wave incidence, the reflection coefficient is reflection-angle dependent and can be calculated if the elastic parameters are given. At any point of the curved interface, based on the tangent plane approximation we can define the local reflection coefficient as \( R(x', \hat{n}, \theta_r) \), where \( x' \) is the location on the surface, \( \hat{n} \) is the outward normal unit vector at the point, and \( \theta_r \) is the reflection angle with respect to the normal. The angle-dependent reflection coefficient defined above is a special kind of *local scattering matrix* defined by the author (Wu and Chen 2002). Here the scattering direction is restricted to only the mirror reflection direction (\( \theta_r = \theta_{\text{in}} \)) in view of the tangent plane approximation. With the definition of the local reflection coefficient \( R \), we write the total field at the point on the boundary as

\[
p(x') = [1 + R(x', \hat{n}, \theta_r)] p^0(x') .
\]

\[
p^\text{ac}(x') = R(x', \hat{n}, \theta_r) p^0(x')
\]

With a plane incident wave
where \( \mathbf{k}_{in} \) is the incident wavenumber vector and \( \mathbf{k}_{in} = (\omega / \nu) \hat{i} \) with \( \omega \) as the frequency, \( \nu \) as the local background velocity, \( \hat{i} \) as the incident unit direction vector. In the following we also use the convention that the source direction \( \hat{s} = -\hat{i} \), which is the direction from point on the surface to the source, opposite of the incident direction. From this definition, we can get the normal derivative of the scattered pressure field at that point (Ogilvy, 1991),

\[
\partial \rho^{sc}(\mathbf{x}') / \partial n = iR(\mathbf{x}')(\mathbf{k}_{in} \cdot \hat{n})\rho^0 = -iR(\mathbf{x}')(\mathbf{k}_s \cdot \hat{n})\rho^0, \tag{10}
\]

where \( \mathbf{k}_s = (\omega / \nu) \hat{s} \). We consider now the plane wave scattering by the local tangent plane, and therefore adopt a plane wave Green’s function

\[
G^{pl}(\mathbf{x}') = \exp[-i\mathbf{k}_g \cdot \mathbf{x}'], \tag{11}
\]

where \( \mathbf{k}_g = (\omega / \nu) \hat{g} \) is the wavenumber vector for the scattered wave with \( \hat{g} \) as the unit vector in the scattering direction. Note that a minus sign is used for the propagator due to the backward direction of the propagation. In a similar way, its gradient is obtained

\[
\partial G^{pl}(\mathbf{x}') / \partial n = -i(\mathbf{k}_g \cdot \hat{n})G^{pl}. \tag{12}
\]

In this case, we can then calculate the scattered field from the representation integral (6) as

\[
p^{sc}(\mathbf{x}) = -i\int_{\Gamma} R(\mathbf{x}', \hat{n}, \theta_r) [(\mathbf{k}_g + \mathbf{k}_s) \cdot \hat{n}] \exp{-i(\mathbf{k}_g + \mathbf{k}_s) \cdot \mathbf{x}'} ds(\mathbf{x}') \tag{13}
\]

We see that the scattered field can be calculated as the contributions from all the secondary sources (equivalent sources by scattering), whose strengths are proportional to the local reflection coefficients. The scattering calculation by (13) is under the physical-optics approximation, which takes into consideration of the diffraction effect of wavefield. The radiation pattern of the secondary sources, i.e. the scattering pattern of the boundary elements, is like that of a dipole, with axis along its normal. The dipole pattern depends only on the exchange wavenumber

\[
\mathbf{k}_\perp = \mathbf{k}_g - \mathbf{k}_{in} = \mathbf{k}_g + \mathbf{k}_s. \tag{14}
\]

However, the reflection strength \( R(\mathbf{x}', \hat{n}, \theta_r) \) is function of lateral wavenumber

\[
\mathbf{k}_\parallel = \mathbf{k}_g + \mathbf{k}_{in} = \mathbf{k}_g - \mathbf{k}_s \tag{15}
\]

through the relation with the reflection angle:

\[
\theta_r = \sin^{-1}[k_g / (2k_0)] \tag{16}
\]

with \( k_\parallel = |k_\perp| \) and \( k_0 = \omega / \nu \). The exchange wavenumber \( k_\perp \) is perpendicular to the local tangent plane, while the lateral wavenumber \( k_\parallel \) is parallel to the tangent plane. Note that the calculation of \( R(\mathbf{x}', \hat{n}, \theta_r) \) uses the mirror reflection of an infinite interface, and therefore, is similar to a geometric-optics approximation. However, the calculation of the scattered field (6) and (13) is a physical-optics approximation and is not limited to the mirror reflection.

**Generalization to surface reflection measurements in heterogeneous media**

In the case of surface reflection profiling on the heterogeneous earth, various one-way propagators can be used as the Green’s function. The source and receiver (geophone) fields can be downward propagated to a level close to the curved interface (Figure 1), and then decomposed...
into local plane waves (Wu and Chen, 2002, 2006; Xie and Wu, 2002). In fact the local plane waves derived by local windowing are beamlets with a dominant direction and certain lobes (ibid). For rigorous treatment, we need to study the interaction between the beamlets and the boundary elements. However, under the same spirit of the Kirchhoff approximation (tangent plane approximation), we will treat the beamlets as plane waves in this paper.

The Kirchhoff integral can be written as

\[
p^{sc}(x_g) = \int_{\Gamma} p^{sc}(x) \frac{\partial}{\partial n} G_M(x_g; x')ds(x') - \int_{\Gamma} G_M(x_g; x') \frac{\partial}{\partial n} p^{sc}(x')ds(x'), \tag{17}
\]

where \(G_M(x_g; x')\) is the Green’s function of modeling which is assumed accurate for the given reference model. In this paper we will use only the primary reflections for the imaging and inversion, and therefore we assume these Green’s functions produce only accurate primary reflections. The incident field is \(G_M(x'; x_g)\). Substitute the expression of the scattered field on the boundary (8) into above equation, resulting in

\[
p^{sc}(x_g) = \int_{\Gamma} R(x', \hat{n}, \theta_r)[G_M(x'; x_g) \frac{\partial}{\partial n} G_M(x_g; x') - G_M(x_g; x') \frac{\partial}{\partial n} G_M(x'; x_g)]ds(x') \tag{18}
\]

The Green’s functions will be decomposed into local plane waves (for local plane wave decomposition, see Appendix A) at depth \(z\),

\[
G_M(x', z; x_g) = \sum_{l} \sum_{j} G_M(\bar{x}_l, \xi_j, z; x_g) b_{jl}(x')
= \sum_{j} e^{i(\xi_j \cdot x' - z)} \sum_{l} g(x' - \bar{x}_l) G_M(\bar{x}_l, \xi_j, z; x_g)
= \sum_{j} e^{i\kappa_j \cdot r} G_M(\xi_j, x', z; x_g)
\tag{19}
\]

where \(G_M(\xi_j, x', z; x_g)\) is the coefficients of a local plane wave component of the Green’s function, \(b_{jl}(x')\) is beamlet atom (element function), and \(g(x - \bar{x}_l)\) is the window function for decomposition (spatial localization), and \(\xi_j\) is the local wavenumber (see Wu and Chen, 2002, 2006). In above equation, \(r' = (x', z' - z)\), and \(\vec{\kappa}_j = (\xi_j, \xi_j)\) is the wavenumber vector, with
\[ \xi_j = \sqrt{(\omega/v)^2 - \xi_j^2} \]
as the vertical wavenumber, for the local plane wave component. It is seen that the original space-domain Green’s function is transformed into a beamlet domain Green’s function (Green’s beamlets), which represents the local plane wave response near the boundary generated by a point source on the acquisition surface. The same decomposition can be applied to the receiver’s Green’s function. Substituting the decomposed Green’s functions and their gradients into (18), we get the similar expression as for the plane wave scattering case:

\[
P^{sc}(x_g, x_s) = -i \sum_j \sum_i \int_{\Gamma} G_M(\xi_j, x', z; x_s) G_M(\xi_j, x', z; x_g) R(x', \hat{n}, \theta)[(\vec{k}^g_j + \vec{k}^s_i) \cdot \hat{n}] \exp[-i(\vec{k}^g_j + \vec{k}^s_i) \cdot r'] ds(x')
\]

In the above equation, the reciprocity of the beamlet Green’s function is used. It means that the beamlet field at depth generated by a point source on the surface is equivalent to the field received by a point receiver on the surface generated by a beamlet source at depth.

**Local Medium Parameters and the angle-dependent reflections**

Unlike the parameters in the Born model, where the elastic or acoustic parameters are explicitly related to the scattered fields, in the boundary scattering model the medium parameters are indirectly related to the scattered waves through the angle-dependent reflection coefficients \( R(x', \hat{n}, \theta) \). For the scattering tomography, we need to relate these coefficients to the local elastic or acoustic parameters.

**Elastic wave case**

We first discuss the case of isotropic elastic media. The angle dependence of the exact Zoeppritz equation is too involved for the purpose of inversion. We will use the Aki-Richards and Shuey’s approximations for our purpose.

Aki-Richards’s approximation (Richards and Frasier, 1976; Aki and Richards, 2002) explicitly relates the angle-dependent reflection coefficients to the parameter (P and S wave velocity, density) perturbations:

\[
R(\theta) = \frac{1}{2} [1 - \frac{4\beta^2}{\alpha^2 \sin^2 \theta}] \frac{\Delta \rho}{\rho} + \frac{1}{2 \cos^2 \theta} \frac{\Delta \alpha}{\alpha} + \frac{[4\beta^2 - \alpha^2 \sin^2 \theta]}{\alpha^2} \frac{\Delta \beta}{\beta}
\]

The original derivation is based on the small perturbation assumption. However, in re-derivation, some restriction can be released (see Appendix B). In fact the approximation is a small scattering-angle approximation, which can be satisfied in either of the two following situations. One is small incident angle for strong contrast media. When the incident angle is small, the scattering angle (refraction angle) is also small even for strong perturbations. The second case is weak perturbation but large incident angle. This will lead to more broad applications of the approximation. However, the approximation is not valid when the incident angle is close to the critical angle.
Shuey’s approximation (Shuey, 1985; Hilterman, 2001) divides the valid region of the relation into near, mid, and far angles, and each region has single parameter dominance:

\[
R(\theta) = \frac{1}{2} \left[ \frac{\Delta \alpha}{\alpha} + \frac{\Delta \rho}{\rho} \right] \left[ 1 - \frac{4 \beta^2}{\alpha^2} \sin^2 \theta \right] + \frac{\Delta \sigma}{1 - \sigma^2} \sin^2 \theta + \frac{1}{2} \frac{\Delta \alpha}{\alpha} \left[ \tan^2 \theta - \frac{4 \beta^2}{\alpha^2} \sin^2 \theta \right]
\]  

(22)

The first term is for the acoustic impedance with valid region 0° - 90°; the second term is for the Poison ratio at mid angle range 15°- 90°; the third term, for P-wave velocity at far-angle 30°- 90° (Hilterman, 2001).

For scattering tomography with small receiver-array aperture, the Shuey’s approximation has some advantages. Since currently the long offset acquisition is still rare, we will use the first two terms in Shuey’s formulas. If the angle coverage is reasonable for the given dip, a two parameter (P-wave impedance and Poison ratio) inversion can be done along the interface.

**Scalar wave case**

In the case of acoustic wave, the reflection coefficient can be expressed as

\[
R = \frac{\rho_2 \cos \theta_i / v_i - \rho_1 \cos \theta_i / v_2}{\rho_2 \cos \theta_i / v_i + \rho_1 \cos \theta_i / v_2}
\]  

(23)

Where \( \theta_i \) and \( \theta_t \) are the incident and transmitted angles respectively. Following Desanto (1992), we can change the reflection coefficient (29) into a form which has the velocity ratio and density ratio as the variables. Using the snell’s law we relate the refracted angle with the incident angle:

\[
\cos \theta_i = (\gamma^2 - \sin^2 \theta_i)^{1/2} / \gamma
\]  

(24)

Where \( \gamma = v_i / v_2 \) is the velocity ratio between the two neighboring layers. Define \( \rho = \rho_1 / \rho_2 \) as the density ratio, Equation (23) can be rewritten as

\[
R = \frac{\rho \cos \theta_i - (\gamma^2 - \sin^2 \theta_i)^{1/2}}{\rho \cos \theta_i + (\gamma^2 - \sin^2 \theta_i)^{1/2}}.
\]  

(25)

From (25) we directly invert for the velocity ratio \( \gamma \):

\[
\gamma = \{[(1 - R)/(1 + R)]^2 \rho^2 \cos^2 \theta_i + \sin^2 \theta_i \}^{1/2}.
\]  

(26)

In the case of constant density, \( \rho = 1 \), and the velocity ratio \( \gamma \) can be inverted directly from the incident angle \( \theta_i \) and the reflection coefficient \( R \).

In some cases, small angle approximation can be applied and the relation (25) and (26) can be simplified to

\[
R(\theta_i) = \frac{1}{2 \cos^2 \theta_i} \frac{\Delta \gamma}{\gamma}
\]  

(27)

And
\[
\frac{\Delta v}{v} = 2R(\theta_r)\cos^2 \theta_r ,
\]

(28)

Where

\[
\Delta v = v_2 - v_1, \quad v = v_2 + v_1 / 2 ,
\]

(29)

and \(\theta\) is the incident or reflection angle.

4. Scattering Tomography for Volume scattering and Boundary Scattering

We will obtain the formulation for both volume scattering and boundary scattering in arbitrarily heterogeneous media. We will see that in generally heterogeneous media, the migration/imaging process is the first step of the tomographic inversion based on scattering theory, followed by a deconvolution filtering which is similar to the amplitude correction in true-amplitude, true-reflection imaging. The final step of parameter estimation is different for volume scattering and boundary scattering.

4.1. Backpropagation and the imaging principle

The traditional diffraction-tomography with a homogeneous reference model applies a filtered backpropagation to the scattered field (data) for the reconstruction of the parameter perturbations in the model space (Devaney, 1982, 1984; Beylkin et al., 1986; Wu and Toksöz, 1987). The backpropagation process is a double-focusing operation, which focus both the waves from the source array and the receiver array to the image point (Wu and Toksöz, 1987). Therefore it is equivalent to applying the imaging condition at the image point in the migration process (Claerbout, 1971, 1985). For homogeneous background with infinite acquisition aperture, the propagation of different plane-wave components is independent of each other, and the deconvolution filter (decon-filter) remains unchanged across the whole image space. Therefore the order of propagation and filtering in wavenumber domain is interchangeable and the filtering is applied to the scattering data before the backpropagation. For heterogeneous backgrounds and/or finite data aperture, this symmetry is broken and the filter is location dependent. In the following, we will first perform the backpropagation to the scattered field for removing the background effect and then apply the space-dependent decon-filter in the local wavenumber domain (or local angle domain). We will see that when the aperture is infinite in a homogeneous background, the process reduces to the filtered backpropagation of the traditional diffraction tomography.

Imaging condition in local angle-domain and local image matrix

The standard imaging condition in the space domain is in the form of cross-correlation,

\[
L(\omega, \mathbf{x}) = 2 \int_{A(x_g)} dx_g G^*_t(\omega, \mathbf{x}; x_g) \cdot \int_{A(x_g)} dx_g \frac{\partial G^*_t(\omega, \mathbf{x}; x_g)}{\partial z} p^w(\omega, x_g; x_s)
\]

(30)

where \(G_t\) is Green’s function used in the imaging process, which could be different from the Green’s function of forward modeling; “ * ” stands for complex conjugate. The inner integral is a back propagation Rayleigh integral and \(A(x_g)\) is the spatial receiver aperture for the given
source. The outer integral is the summation over all the sources and \( A(x_i) \) is source aperture. This is the double focusing for a single-frequency wavefield. In migration imaging, the final image is obtained by summing up images of all the frequencies, a double-focusing in time-domain. In scattering tomography, both single-frequency tomography and multi-frequency tomography can be implemented.

In order to obtain the local angle-spectra of an image field, imaging condition has been extended from space domain to the space-angle-domain (beamlet domain) (Wu and Chen, 2002, 2004). Then the image function is no longer a scalar value and becomes a matrix: LIM (local image matrix) \( L(x, k_s, k_g) \), where \( x = (x, z) \) is the position vector at depth \( z \); \( k_s \) and \( k_g \) are the source and receiving wavenumbers, respectively. To simplify the notations, we omit the frequency dependence in the LIM and the following derivations. When necessary, we will specify the frequency dependence again explicitly. Note that the source direction is defined as the direction from the image point to the source on the surface, and is opposite to the incident direction. The new imaging condition (for a single frequency) in the space-angle domain is

\[
L(x, k_s, k_g) = 2 \int_{A} dx_s G'_f(x, k_s; x_i) \cdot \int_{A} dx_g \frac{\partial G'_f(x, k_g; x_s)}{\partial z} p_{ac}(x_s; x_i)
\]

(31)

where \( G'_f(x, k_s; x_i) \) is the incident beamlet at the image point generated by a point source at \( x_i \) on the surface, and \( G'_f(x, k_g; x_i) \), the outgoing (scattered) beamlet at the imaging point received by a point receiver at \( x_g \) on the surface. If the Green’s functions used in the inversion are exactly the same Green’s functions as those in the forward modeling (in the acquisition process), or their phase information are exact, the LIM will be a real number matrix. However, due the approximations in Green’s function and velocity model inaccuracy, the LIM in general is a complex matrix:

\[
L(x, k_s, k_g) = [A_L \exp(i\Phi_L)],
\]

(32)

where \( A_L = A_L(x, k_s, k_g) \) is the amplitude of matrix element and \( \Phi_L = \Phi(x, k_s, k_g) \) is the phase of matrix element. For an inaccurate background velocity model, the phase \( \Phi_L \) is different for different pairs of \( (k_s, k_g) \), resulting in incomplete focusing. These phase residues can cause the blurriness of the final image and produce artifacts and coherent noises. However, the image phase-residue matrix can also be used to update the velocity model during an iterative process.

**4.2. Deconvolution filtering in local wavenumber domain**

We first treat the scalar wave case. The modeling and imaging processes can be written into operator forms:

\[
U(\omega, x_g, x_i) = F(\omega, x_g, x_i | x_0)S(x_0)
\]

(33)

\[
I(x) = B(x | \omega, x_g, x_i)U(\omega, x_i, x_g)
\]

(34)

where \( F \) is the acquisition (modeling) operator which maps the model \( S \) into the data set \( U \); while \( B \) is the imaging operator which invert the data \( U \) into the image \( I \), \( x_0 \) is the scattering point in the model space, and \( x \) is the location of the image point in the imaging process. Substitute (33) into (34) we have

\[
I(x) = B(x | \omega, x_g, x_i)F(\omega, x_g, x_i | x_0)S(x_0).
\]

(35)
The resolution operator (matrix) is obtained as

$$R(x, x_0) = B(x | \omega, x_s, x_g)F(\omega, x_g, x_s | x_0)$$  \hspace{1cm} (36)$$

If the imaging operator is exactly the inverse operator of the modeling operator, the resolution operator will be an identity operator. For most cases, the resolution matrix is not an identity matrix and the spreading of the matrix elements along the diagonal give some quantitative measure of the parameter resolution of the imaging/inversion. If the resolution matrix can be calculated, the true model can be obtained by deconvoluting the image field with the resolution matrix (with some regularization). However, the calculation of resolution operator and volume deconvolution in space-domain is intractable. We will formulate the process of decon-filtering in the local wavenumber domain, i.e. the beamlet domain. In the following, we treat the volume scattering and boundary scattering separately.

### 4.3. Scattering tomography for volume scattering based on the Born model

In the following we will derive the resolution operator for the imaging process based on the Born model for volume perturbations.

In order to be symmetric for the source array focusing and receiver array focusing and compare later with the classic diffraction tomography, we slightly modify the imaging condition (30) into

$$L(\omega, x) = 4 \int_{\Delta_s} dx_s \frac{\partial G_s^*(\omega, x_s, x_g)}{\partial z} \int_{\Delta_g} dx_g \frac{\partial G_g^*(\omega, x_g, x_s)}{\partial z} p^\omega(\omega, x_g, x_s)$$  \hspace{1cm} (37)$$

In this imaging condition, we propagate the source field as if from boundary elements instead of volume elements (isotropic source). Later we will see the symmetry and convenience of this modification. After decomposing the Green’s functions into the local wavenumber-domain (equation A8 in Appendix A) at a level $z$ near the image point $(x', z')$, and substituting into (37), the image matrix at $x'$ becomes

$$L(x', \mathbf{k}_s, \mathbf{k}_g) = \exp\{-i(\mathbf{k}_g + \mathbf{k}_s) \cdot x'\} 4 \int_{\Delta_s} dx_s G_s^*(x, \mathbf{k}_s, x_g) \int_{\Delta_g} dx_g G_g^*(x, \mathbf{k}_g, x_s) p^\omega(x_g, x_s)$$  \hspace{1cm} (38)$$

where $x' = (x', z' - z)$ and $\mathbf{k}_s = (\xi_s, \eta_s), \mathbf{k}_g = (\xi_g, \eta_g)$, with $\xi$ and $\eta$ as the corresponding local horizontal and vertical wavenumbers respectively. From Equation (38), we see that from a pair of local incident-scattering angles we can only detect the local spectral component at $\mathbf{k} = \mathbf{k}_s + \mathbf{k}_g = \mathbf{k}_s$. Set $z' = z$, equation (38) can be written into another form

$$L(x', \mathbf{k} = \mathbf{k}_s) = -4 \xi_s \eta_s \exp\{-i(\xi_g + \xi_s) \cdot x'\} \int_{\Delta_s} dx_s G_s^*(x', \mathbf{k}_s, x_g)$$  \hspace{1cm} (39)$$

Figure 2 shows the definition of local wavenumbers and the domain of spectral coverage (domain of integration) for the Born model. We will later compare it with the that for the case of boundary scattering model.

With multi-frequency imaging, the spectral coverage can be expanded and the local image matrix becomes
\[ I(x', \bar{K}) = \int_{A} d\omega L(\omega, x', \bar{K} = \bar{K}_\perp). \] (40)

Figure 2: The definition of local wavenumbers and the domain of spectral coverage for the Born model

The angular spectra of local images are usually highly non-uniform due to the acquisition geometry, limited data aperture and the influence of overburden structures. In order to reconstruct the true parameter of the perturbation function (the object function), we need to deconvolute the local image field with the resolving kernel.

The kernels for the resolution operator (resolving kernel) (Backus and Gilbert, 1970; Tarantola, 1987, 2005) is obtained based on (36), (37) and (4). In order to being generally applicable, we start from the multi-frequency formulation:

\[ \Re(x; x_0) = -4 \int d\omega k^2 \int_{A} dx_x \frac{\partial G_x^*(\omega, x; x_x)}{\partial z} G_M(\omega, x_0; x_x) \int_{A} dx_g \frac{\partial G_g^*(\omega, x; x_g)}{\partial z} G_M(\omega, x_0; x_g) \] (41)

For detailed derivation of the resolution operator and its kernel, see Wu et al. (2006). The resolving kernel is also called the point spreading function (PSF) for the imaging system which includes both the acquisition (modeling) and inversion (imaging) processes. We will formulate the process in the local wavenumber domain, i.e. the beamlet domain. We perform local 3D Fourier transform (local wavenumber-domain decomposition) on \( \Re(x, x_0) \) with coordinate center at \( x_0 \), resulting in the expression of PSF in local wavenumber domain (Wu et al., 2006) (see also Appendix C):

\[ \Re(k = k_\perp; x_0) = -2 \int_{A} d\omega k^2 \int_{A} dx_x \frac{\partial G_x^*(\omega, x_0, \bar{K}_\perp; x_x)}{\partial z} G_M(\omega, x_0; x_x) B(\omega, x'; \bar{K}_\perp) \] (42)

where \( B(\omega, x'; \bar{K}_\perp) \) is the backpropagation integral of the modeling Green’s function,
\[
B_r(\mathbf{x}', \bar{\mathbf{k}}_g) = 2 \int_{\mathbb{R}^d} dx_g \frac{\partial G_i^r(\mathbf{x}', \bar{\mathbf{k}}_g; x_g)}{\partial z} G_M(\mathbf{x}', x_g).
\]  

(43)

The image field I is seen from (35) a convolution of the resolution operator R with the model S. Therefore, the correct object function is obtained by deconvoluting the image with the resolving kernel. In the local wavenumber domain, the decon-filter is a division of (40) by (42):

\[
O(\mathbf{x}, \bar{\mathbf{K}}) = I(\mathbf{x}, \bar{\mathbf{k}}_\perp) \cdot \Re(\bar{\mathbf{k}}_\perp; \mathbf{x})
\]

(44)

The local image in the space-domain can be reconstructed by the inverse beamlet transform,

\[
O(\mathbf{x}') = \iint d\bar{\mathbf{K}} O(\mathbf{x}', \bar{\mathbf{K}}) = \int_{A_\vartheta} d\theta \int_{A_k} dK [KO(\mathbf{x}', K, \vartheta)]
\]

(45)

where \( O(\mathbf{x}', K, \vartheta) \) is local angular spectrum of the object function and \( A_\vartheta, A_k \) are the angular and radial coverage of the spectrum.

The above approach is to perform the decon-filtering to the final multi-frequency image. An alternative approach is to filter the images in the local wavenumber domain for each single frequency, and obtain the final object spectrum by averaging over the single-frequency object spectra. For a single-frequency, the reconstruction (44) becomes

\[
O(\mathbf{x}') = \iint d\bar{\mathbf{K}} O(\mathbf{x}', \bar{\mathbf{K}}) = \iint d\bar{\mathbf{K}}_x d\bar{\mathbf{K}}_z O(\mathbf{x}', \bar{\mathbf{K}}_\perp)
\]

\[
= \iint d\bar{\xi}_x d\bar{\xi}_z J(\mathbf{K}_x, \mathbf{K}_z; \bar{\xi}_x, \bar{\xi}_z) O(\mathbf{x}', \bar{\mathbf{K}}_\perp)
\]

\[
= \iint d\bar{\xi}_x d\bar{\xi}_z \left| \frac{\bar{\xi}_x \bar{\xi}_z - \bar{\xi}_g \bar{\xi}_g}{\bar{\xi}_g \bar{\xi}_g} \right| \frac{L(x', \bar{\mathbf{k}}_\perp)}{\Re(\bar{\mathbf{k}}_\perp; \mathbf{x}')} \]

(46)

which is seen to be a backpropagation plus filtering process. In (46) \( J(\mathbf{K}_x, \mathbf{K}_z; \bar{\xi}_x, \bar{\xi}_z) \) is the Jacobien of the coordinate transform from \( (\bar{\xi}_x, \bar{\xi}_z) \) to \( (\mathbf{K}_x, \mathbf{K}_z) \).

Relation with the classic diffraction tomography in a homogeneous background

In a homogeneous media with infinite aperture of acquisition, the local wavenumber becomes global and the beamlet decomposition is replaced by the global Fourier transform. Then the Green’s function in (39) becomes

\[
G_i(\mathbf{k}_z; x_g) = \frac{i}{2} \exp \left( \frac{i \zeta_s z}{\zeta_s} \right) \exp \left( \frac{i \xi_z x_g}{\xi_z} \right).
\]

(47)

Substituting the Green’s function into (39), resulting in

\[
L(z', \bar{\mathbf{k}} = \bar{\mathbf{k}}_\perp) = e^{-i(\zeta_z + \xi_z)z'} \int dx_g e^{-i \zeta_s x_g} \int dx_g e^{-i \xi_z x_g} p^\infty(x_g; x_g)
\]

\[
= e^{-i(\zeta_z + \xi_z)z'} p^\infty(\zeta_z; \xi_z)
\]

(48)

This is a simple backpropagation of the scattered field in the wavenumber domain. In a similar way, the decon-filter in (42) for a single-frequency becomes
Scattering Tomography

\[ \mathcal{R}(\mathbf{k}_\perp; \mathbf{x}_0) = k^2 \int dx_x e^{-i\mathbf{k}_\perp \cdot \mathbf{x}_x} G_M(\omega, \mathbf{x}_0; \mathbf{x}_x) \int dx_y e^{-i\mathbf{k}_\perp \cdot \mathbf{x}_y} G_M(\omega, \mathbf{x}_y; \mathbf{x}_0) \]
\[ = \frac{k^2}{\xi_s \xi_g}. \quad (49) \]

The tomographic inversion in wavenumber domain becomes

\[ O(\mathbf{k}_\perp) = \frac{L(\mathbf{k}_\perp)}{\mathcal{R}(\mathbf{k}_\perp)} \]
\[ = e^{-i(\xi_s + \xi_g) \cdot \mathbf{k}_\perp} \frac{\xi_s \xi_g}{k^2} p^{sc}(\xi_g, \xi_s). \quad (50) \]

Transforming back to the space-domain, we get the tomographic image of the object function:

\[ O(\mathbf{x}') = \int d\mathbf{k} O(\mathbf{k}_\perp) = \int dK_x dK_z O(\mathbf{k}_\perp) \]
\[ = \int d\xi_x d\xi_y I(K_x, K_z; \xi_x, \xi_y) e^{-i(\xi_s + \xi_g) \cdot \xi} \frac{\xi_s \xi_g}{k^2} p^{sc}(\xi_g, \xi_s). \quad (51) \]

The above formulation is exactly the filtered backpropagation of classic diffraction tomography in homogeneous backgrounds for surface reflection data with an infinite aperture (Devaney, 1982, 1984; Wu and Toksöz, 1987). In the more general case of heterogeneous backgrounds, as we proposed above, the process is a backpropagation plus filtering (deconvolution) in the local angle-domain.

**Scattering tomography and true-amplitude imaging**

Normally the image matrix is a complex matrix with phase residual as a function of local wavenumbers. If we assume that the background velocity model is exact and the propagators do not have phase errors, our concern can concentrate to the angular spectra of amplitude of the image field. Now we discuss the relation of scattering tomography with the true-amplitude imaging with amplitude corrections in the local angle domain (Wu et al., 2004; Wu and Luo, 2005; Cao and Wu, 2005).

Consider only the amplitude correction in the decon-filter calculation in (42), we have

\[ F_i(\mathbf{k}_\perp, \mathbf{x}') = |\mathcal{R}(\mathbf{k}_\perp; \mathbf{x}')| \]
\[ = -2 \int d\mathbf{k} d\omega k^2 \int d\mathbf{x}_x \left| \frac{\partial G_i^*(\omega, \mathbf{x}', \mathbf{k}_\perp; \mathbf{x}_x)}{\partial \xi} \right| \left| G_M(\omega, \mathbf{x}_x; \mathbf{x}') \right| \left| B_i(\mathbf{x}', \mathbf{k}_g) \right| \quad (52) \]

where \( B_i(\mathbf{x}', \mathbf{k}_g) \) is the backpropagation integral (43).

As we have seen, in the calculation of decon-filter, we need to use the Green’s function for the modeling. Sometimes the exact Green’s function, i.e. the pulse response of the real medium, is difficult to know or computationally expensive if known. If we make the following approximation
in above equations, the decon-filter in (52) reduces to the amplitude correction factor in true-amplitude, true-reflection imaging (ibid). The approximation (53) makes the calculation of amplitude correction for compensating aperture effects into an efficient procedure. The resolving kernel in the reconstruction formula (46) then can be replaced by the amplitude correction factor (52).

4.4. Scattering tomography for boundary scattering based on the Kirchhoff model

Resolving Kernel and True-Reflection Recovery

The resolving kernel for the boundary scattering model is different from that of the volume scattering model. In the latter case the goal of inversion is to reconstruct the spectrum of the object function; while the former is to recover the reflection coefficients \( R(x', \hat{n}, \theta) \). We see from equation (20) that under the Kirchhoff modeling, the scattered field is the summation of the contributions from all the boundary elements. The interaction between the elements is neglected. From the imaging principle (31) and the boundary scattering model (20), we obtain the kernel for a single-frequency imaging:

\[
G_M(x'; x_g) = G_I(x'; x_g)
\]

(53)

where the backpropagation integral \( B_I(x', \hat{k}_g) \) is defined by (43) with the replacement of \( G_M(x'; x_g) \) by \( G_I(x', \hat{k}_g; x_g) \). For a given \( \theta_r \), the corresponding magnitude of \( \hat{k} \) is fixed as \( 2k \sin \theta_r \), and the resolving kernel defines how the imaging system maps a single dip into a distribution in the local angle-domain of the image. Comparing (54) with the resolving kernel for volume scattering (42), we can see the difference between the two cases. The spectral coverage for boundary scattering is defined on a h-f asymptotic sphere (a circle in the 2D case). The resolution is only for the dip-direction and do not have resolution in the magnitude of the spectral component. This is due to the assumption of parameter discontinuity at the boundary reflection point. Figure 3 shows the definition of the local wavenumbers and the asymptotic sphere for a single-frequency. For a single-dip boundary element with normal \( \hat{n} \), the spreading pattern in the asymptotic circle is a narrow lobe. We can obtain the dip by picking the peak value of the lobe. The range of detectable dip-angles can be studied by the directional illumination analysis. Knowing the dip, the decon-filtering in the local wavenumber domain can be implemented to recover the true-reflection in dip-angle domain

\[
R(x', \hat{n}, \theta_r) = \sum_{\hat{k}} \left\{ L(x, \hat{k}_g, \hat{k}_g') / \mathcal{R}(\hat{k}_g, \hat{k}_g'; x', \hat{n}, \theta_r) \right\}.
\]

(55)

For multi-frequency imaging,

\[
R(x', \hat{n}, \theta_r) = \frac{1}{N_f} \int_{A_f} d\omega \sum_{\hat{k}} \left\{ L(\omega, x', \hat{k}_g, \hat{k}_g') / \mathcal{R}(\omega, \hat{k}_g, \hat{k}_g'; x', \hat{n}, \theta_r) \right\}
\]

(56)

Where \( N_f \) is a normalization factor for the integration over the frequency band \( A_f \). Therefore the reconstructed reflection coefficient in (56) can be looked as an averaged value over the frequency band. The filtering and summation over multi-frequencies can be looked as an averaging and spectrum-whitening process in the reconstruction.
The procedure for reflectivity tomography can be summarized as the follows. First is the backpropagation and imaging in local angle domain for each frequency. The local dip at each image point is estimated by a dip-scan at the local image matrix (LIM). Then the decon-filter (54) is applied to the LIM to reconstruct the local reflection coefficient (55). The final step is the summation over the frequency band. Comparing with the tomographic reconstruction (46) for the Born model, we can see the significant difference between these two reconstructions.

**Relation to the true-reflection imaging with amplitude correction**

If the velocity model is known and the propagator phase-error can be neglected, only the image amplitude needs to be corrected to recover the true-reflection. Then the resolving kernel (54) becomes a purely amplitude correction factor:

\[
 F_a(x', k_s, k_g) = \left| R(\vec{k}_s, \vec{k}_g; x', \hat{n}, \theta_r) \right| \\
 = \left| \int_{A} dx_s [(\vec{k}_g + \vec{k}_s) \cdot \hat{n}] \left| G_s(x', \vec{k}_s; x_s) \right| \left| G_M(x', \vec{k}_s; x_s) \right| B_s(x', \vec{k}_g) \right|. 
\]  
(57)

Then the amplitudes of the LIM can be used to recover the true-reflection:

\[
 R(x', \hat{n}, \theta_r) = A_f(x', k_s, k_g) / F_a(x', k_s, k_g) 
\]  
(58)

Where \( A_f(x', k_s, k_g) \) are amplitude elements of the local image matrix (32). The amplitude factor defined by (57) is the same as that derived by Wu et al. (2004) for true-reflection imaging. For the purpose of efficient implementation, two further approximations were adopted in their applications. First is the replacement of \( G_M \) with \( G_s \), and the second the use of frequency-independent amplitude factor, i.e. the assumption that

\[
 F_a(\omega) = F_a(\omega_0), 
\]  
(59)

where \( \omega_0 \) is the dominant frequency of the source wavelet. The latter approximation makes the procedure very efficient in the acquisition-aperture correction for true-amplitude imaging, and can keep the shape of the wavelet in the final image. However, from the viewpoint of tomographic inversion of reflectivity, the image amplitudes may not represent the true-reflection coefficients. On the other hand, if we correct the amplitudes with (58) for each frequency and
sum up over the frequency band using the multi-frequency true-reflection recovery of (56), the resulted images will resemble spectrum-whitened reflectivity images. Then (56) becomes

\[
R(\mathbf{x}', \mathbf{n}, \mathbf{\theta}_r) = \frac{1}{N_f} \int_{\Delta f} d\omega \sum_{k} \{ A_k(\omega, \mathbf{x}, \mathbf{k}_\perp, \mathbf{k}_\parallel) / F_a(\omega, \mathbf{x}', \mathbf{k}_\perp, \mathbf{k}_\parallel) \} .
\]

(60)

For boundary scattering model with tangent-plane approximation, true-reflection imaging is a practical and efficient way of implementing the scattering tomography. As we discussed in the introduction, models of this kind can be decomposed into a smooth background part and a discontinuity part composed of sharp interfaces. Since the amplitude variation of wavefield is relatively slow in the smooth background, the amplitude correction will have enough accuracy even if the phase error in the local image matrix caused by inaccurate velocity models or/and propagator phase-errors. The phase information in the LIM can be utilized separately for the background parameter estimation where the forward scattering is dominant. Here we first discuss the calculations of the resolving kernel (54) and the amplitude correction factor(57).

If we need to calculate both the phase and amplitude of the integral, we need the accurate information of the velocity model of the medium. If we do not have the exact velocity model, we can use a trial model and the phase residuals in the LIM can help to update the model. We will discuss this more later. However, since the amplitude varies much slower than the phase, we can perform the amplitude corrections even with even inaccurate velocity models. In the following we discuss the calculation of the magnitude of the backpropagation integral for purpose of amplitude correction. The back propagation integral in (43) represents a refocusing process of the scattered waves received by the receiver array on the surface. This integral involves a beamlet forward propagation (spreading) and a back propagation (refocusing) with limited wave front aperture. For large enough aperture, the aperture lobe is usually narrower than the beam width. From energy (amplitude) point of view, we can use a “transparent” Green’s function, or precisely defined, energy-conserved Green’s function \( G_E \) for back propagation. In this way, all the energy loss during refocusing, such as boundary reflection, P-S conversion, scattering and anelastic attenuation, will be neglected, so that we can conserve all the energy collected by the receiver array to the maximum degree. Assuming we use \( G_E \) for \( G_f \), then energy conservation exists

\[
| B_A(\mathbf{x}', \mathbf{k}_g) |^2 = \int_{\Delta x_g} dx_g | G_M(\mathbf{x}', \mathbf{k}_g; x_g) |^2,
\]

except for some edge beamlets. By reciprocity, the beamlet Green’s function in (61) can be calculated as radiated from a point source on surface at \( x_g \) and received by beamlet antenna at \( \mathbf{x}' \) with the same angle and beam-width. Therefore the receiver aperture effect can be calculated by similar procedure as the acquisition aperture response using the Gabor transform (Gabor-Daubechies frame decomposition) (Wu and Chen, 2002a,b, 2006). For details for aperture correction, see Wu et al. (2004). The approximation made in (61) increases the computation efficiency considerably.

**Tomographic inversion for velocity contrast**

If we use the weak scattering approximation (28) in the boundary scattering model (20), we have an explicit relation between the data (scattered field) and the velocity perturbations:
\[ p^{\tau}(x', s') = -i \sum_i \sum_j \int G_M(\xi_i', x', z; x_i) G_M(\xi_j', x', z; x_j) \exp{\frac{\theta}{2 \cos^2 \theta_r} \cdot \hat{\mathbf{K}}_j \cdot \hat{n}} \exp{\{-\hat{\mathbf{K}}_j \cdot \mathbf{r}'\}} ds(\mathbf{x}') \]  

Where

\[ \varepsilon_v = \Delta v / v = (v_i - v) / v \]

as the relative velocity perturbation and \( v_i \) is the velocity below the interface. The resolving kernel then becomes

\[ \Re(\hat{\mathbf{K}}_j; x', \varepsilon_v, \hat{n}) = \int dx_x [\hat{\mathbf{K}}_j \cdot \hat{n}] \frac{2k^2}{k_j^2} G_j(x, \hat{\mathbf{K}}_j; x) G_M(\mathbf{x}', \hat{\mathbf{K}}_j; x_j) B_j(x', \hat{\mathbf{K}}_g). \]  

The corresponding amplitude factor is

\[ F_a(\mathbf{x}', \hat{\mathbf{K}}_j, \hat{n}) = \left| \Re(\hat{\mathbf{K}}_j; x', \varepsilon_v, \hat{n}) \right| \]

Similar to (55), the reconstruction of velocity perturbations can be formulated as

\[ \varepsilon_v(\mathbf{x'}, \hat{n}) = \frac{\sum_{k_j} L(\mathbf{x}, \hat{\mathbf{K}}_j, \hat{\mathbf{K}}_||) / \sum_{k_j} \Re(\hat{\mathbf{K}}_j, \hat{n}, \mathbf{x}'; \theta)}{\sum_{k_j} L(\mathbf{x}, \hat{\mathbf{K}}_j, \hat{\mathbf{K}}_||).} \]  

Where \( L_{CDI}(\mathbf{x}, \hat{\mathbf{K}}_j, \hat{\mathbf{K}}_||) \) are the common dip-angle image (CDI) gathers. For amplitude correction, the resolving kernel is replaced by the amplitude factor in (66). The deconvolution or amplitude correction has to be conducted in the dip-angle domain, since the acquisition-aperture effect is highly dip-dependent.

**Velocity updating by transmission tomography for boundary scattering model**

Usually phase (travel-time) information is more stable than the amplitudes, and travel-time can be directly correlated with the interval velocity between interfaces. We can use the phase mismatch information to estimate or update the interval velocities. With the updated velocity (here the P velocity), we can correct the aperture effect to get the true reflection amplitudes (see next section). The reflection amplitude is most sensitive to the impedance contrast between the two adjacent layers and a local inversion will provide estimation on the impedance. If the near-surface density can be obtained, then the density of the second layer can be estimated from the impedance contrast and the transmission tomography. Layer-by-layer iterative procedure can be set up to update the parameter estimation.

At the current interface of the reference model (from the previous iteration or the priory velocity model), phase (or travel-time) errors in local angle domain can be obtained from the local image matrix (32). The phase error \( \Phi_L \) must be phase unwrapped, if Rytov approximation is used in tomographic inversion (Wu and Toksöz, 1987; Lo et al., 1988). The other approach is to get the local travel-time error matrix (local \( \tau \) matrix) by local Fourier transform. Instead of stacking the images for all the frequencies, a local Fourier Transform is performed:

\[ L(\mathbf{x}, \theta, \theta_g) = \int_{a_0}^{a_0 + a_0^\alpha} d\omega L(\mathbf{x}, k_s, k_g) \exp{-i\omega(t-t_0)} \]  

where \( a_0 \) is the wavelength and \( \omega \) is the frequency.
Then the complex local image matrix (32) becomes
\[ L(x, k_s, k_g) = [A_L \exp(i\Phi_L)] \]
where,
\[ \tau_L(x', k_s, k_g) = \Phi_L(x', k_s, k_g) / \omega \]
is an element of the local tau-matrix.

Knowing the time-delay \( \tau_L \) in the local angle domain, a back-projection scheme has to be set up for the inversion. Since the inter-layer structure is rather smooth, Rytov approximation can be used for the backpropagation (Devaney, 1984; Wu and Toksöz, 1987; Lo et al., 1988). If the velocity structure is very smooth, a ray back-projection is the simplest updating method. In this case, \( \tau(x', k_s, k_g) \) will be back-projected using ray-tracing to the round-trip path.

For more general case where both the locations of curved interfaces and the interval velocities have errors, a global diffraction tomography can be set up for the whole medium or up to certain depth where a rather flat reference reflecting-interface can be found. The local tau-matrices along the reference interface may serve as the differential travel-time data with respect to the reference model for the transmission experiments. These travel-time errors will be back-projected to the model space with wave backpropagation, similar to the procedure in transmission diffraction tomography.

5. Conclusion

Tomographic inversions can be summarized as a process of backpropagation plus filtering in the local angle domain. The backpropagation is a doubly focusing process, similar to the imaging principle in migration/imaging. The filtering is a deconvolution in the local angle domain, and is different for the Born model of volume scattering and the Kirchhoff model of boundary scattering. The decon-filtering formulation derived in this paper is for arbitrarily heterogeneous media, including those with sharp boundaries, with limited acquisition aperture. For the Born model in a homogeneous background with infinite acquisition aperture, the formulation reduces to the classic diffraction tomography. In the case of boundary scattering model, if the velocity model is accurate and the propagators do not have phase error, the decon-filtering will be similar to the amplitude correction in the true-reflection imaging. It is shown that under weak scattering approximation, the true-reflection images for the boundary scattering model can be formulated into a direct inversion of local velocity contrast along the boundaries.

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Appendix A: Beamlet decomposition of Green’s functions
Wave fields in the frequency-space domain can be represented as \( u(x_T, z) \) for a given frequency \( \omega \), where \( x_T \) is the horizontal position vector \( x_T = (x, y) \) in the 3D case, and \( x_T = x \) in the 2D case, and \( z \) is the depth. For the sake of simplicity, we omit the parameter \( \omega \) in the wave field representation. Equivalently, it can be represented in the frequency-wavenumber domain as \( u(k_T, z) \), where \( k_T = (k_x, k_y) \) is the horizontal or transverse wavenumber vector. However, these two domains cannot coexist simultaneously. In space domain the space localization is perfect. That means you can specify precisely the phase and amplitude of a wave field at any point; but you can not specify the propagation direction at all. On the other hand, in wavenumber domain you can have perfect direction localization, but no space localization at all. This is one of the most important differences of the traditional wave field representation (either space-domain or wavenumber-domain) from the ray representation (asymptotic approximations).

Recent progress in beamlet decomposition of wave field (Steinberg, 1993; Wu et al., 2000; Wu and Chen, 2001, 2002a, b, 2006; Xie and Wu, 2002) provides a basis for localizing the wave field in both space and direction simultaneously. Beamlet transform uses translated windows for spatial localization and harmonic modulations for directional localization. The sizes of spatial and directional localizations cannot be arbitrarily small simultaneously and must satisfy the Heisenberg uncertainty principle. This is an important difference between the wave theory in beamlet-domain and the ray theory, similar to the difference between the quantum mechanics and the classical mechanics.

The decomposition element of the Gabor decomposition (Gabor-Daubechies transform) is a Gaussian windowed exponential harmonics

\[
g_{mn}(x) = e^{i \Delta x_n} g(x - n \Delta_x),
\]

where \( g(x) \) is a window function, \( \Delta_x \) and \( \Delta_\xi \) are the space and wavenumber sampling intervals respectively, \( n \Delta_x \) and \( m \Delta_\xi \) are the corresponding space-wavenumber domain loci of \( g_{mn}(x) \). In order to have efficient decomposition and reconstruction, the transform is based on the Gabor-Daubechies frame theory (for a summary see the Appendix of Wu and Chen, 2006). For a detailed mathematical treatment see Daubechies 1992. Here we give only the necessary definition and physical explanation.

A G-D frame is a type of windowed Fourier frame (Daubechies, 1992) using the Gaussian window function. Frame decomposition is not orthogonal and therefore has redundancy in the representation. Gabor (1946) originally proposed a decomposition using the critical sampling \( \Delta_x \Delta_\xi = 2\pi \) in the time-frequency domain. Daubechies (1990) has proved that the reconstruction using critical sampling is unstable, and for stable reconstruction, over-sampling \((\Delta_x \Delta_\xi < 2\pi) \) must hold, where \( \Delta_x \Delta_\xi \) measures the size of decomposition atoms. The necessary and sufficient conditions for the stable reconstruction have been derived based on the frame theory (Daubechies, 1990; 1992). For 2D cases, beamlet decomposition of a wavefield at depth \( z \) using the G-D frame can be expressed as (Wu and Chen, 2002b, 2006):

\[
u(x, z, \omega) = \sum_m \sum_n \langle u, \tilde{g}_{mn} \rangle g_{mn}(x) \\
= \sum_m \sum_n \hat{u}_z(\tilde{x}_n, \tilde{z}_m, \omega) \tilde{g}_{mn}(x),
\]

where \( \hat{u}_z(\tilde{x}_n, \tilde{z}_m, \omega) \) are the beamlet coefficients, \( \omega \) is the circular frequency, \( g_{mn} \) and \( \tilde{g}_{mn} \) are the G-D frame atoms and dual frame atoms respectively:
where $\bar{x}_n = n\Delta_x$, $\bar{\xi}_m = m\Delta_\xi$ with $\Delta_x \Delta_\xi < 2\pi$ are the $n$th window location and the $m$th local wavenumber position respectively. $g(x)$ is a Gaussian window function with $\tilde{g}(x)$ as its dual window function. The dual window function can be calculated by pseudo-inversion of the original window function (Mallat, 1998; Qian and Chen, 1996; Wu and Chen, 2001). For wave field propagation in beamlet domain, we decompose the wave field with the G-D frame such that its dual window function is very close to the Gaussian window function. This is the case of tight frame (see Daubechies, 1992). We see that for wave field decompositions, each beamlet (in this case a G-D frame atom) is a windowed plane wave that has both space localization ($\bar{x}_n$) and direction localization ($\bar{\xi}_m$). Due to the uncertainty principle $\Delta_x \Delta_\xi < 2\pi$, the local parameters in beamlet domain $\bar{x}_n$ and $\bar{\xi}_m$ are different from those in the space-wavenumber domain $x$ and $\xi$.

The beamlet position is only specified as a local window centered at $\bar{x}$, and $\bar{\xi}$ specifies only the lobe direction of the beamlet centered at $\bar{\xi}$ in the wavenumber domain. The Gabor beamlet (Gaussian beamlet) has a smooth lobe without side lobes. The width of the lobe is inversely proportional to the width of the spatial window. Beamlets can be propagated by propagators in beamlet domain and then form images by applying the imaging condition in either the beamlet domain or the space domain (Wu et al., 2000; Wu and Chen, 2001). At each step, the wave field in the space domain can be reconstructed by summing up the contributions from the beamlets (inverse beamlet transform):

$$u(x, z, \omega) = \sum_{j} \sum_{l} \hat{u}_{z}(\bar{x}_l, \bar{\xi}_j, \omega) g_{j,l}(x)$$

$$= \sum_{l} e^{i\bar{\xi}_l \omega} \sum_{l} g(x - \bar{x}_l) \hat{u}_{z}(\bar{x}_l, \bar{\xi}_j, \omega)$$

(A4)

It can be seen from equation (4) that at each space location the field can be recovered by superposing the contributions of all the windowed plane waves (beamlets) from all the neighboring windows. Due to the nature of Gaussian windows, for each $x$, the field is mainly controlled by the beamlets in a few neighboring windows. If we sum up the contributions of all the neighboring windows for the same local wavenumber,

$$u(x, z, \bar{\xi}_j, \omega) = e^{i\bar{\xi}_j \omega} \sum_{l} g(x - \bar{x}_l) \hat{u}_{z}(\bar{x}_l, \bar{\xi}_j, \omega) = e^{i\bar{\xi}_j \omega} \bar{u}(x, z, \bar{\xi}_j, \omega)$$

(A5)

the resulted $u(x, z, \bar{\xi}_j, \omega)$ is a local plane wave and $\bar{u}(x, z, \bar{\xi}_j, \omega)$ is its coefficient. Therefore the local plane wave for location $x$ is an average beamlet over its neighboring windows. For the local plane wave of local wavenumber $\bar{\xi}_j$, the corresponding propagating angle is

$$\bar{\theta}_j = \sin^{-1}(\bar{\xi}_j, v(x, z)/\omega),$$

(A6)

where $\bar{\theta}_j$ is the local incident angle with respect to the vertical, and $v(x, z)$ is the wave velocity at $(x, z)$. 
For a Green’s function $G(x, z; x_s)$, which is the wavefield received at $(x, z)$, excited by a point source at $x_s$, a similar decomposition can be obtained at the receiving level $z$:

$$G(x, z; x_s) = \sum_j e^{ie_j \cdot (x-x_s)} G(\xi_j, x, z; x_s)$$  \hspace{1cm} (A7)

To simplify the notation, we omit the bar on the beamlet coefficient of the Green’s function. With this notation, $G(\xi_j, x, z; x_s)$ is only the local plane-wave coefficient without the plane wave phase factor, which is in consistent with the convention of Fourier transform. Given the beamlet-domain Green’s function at level $z$, we can express the field at any point near the decomposition level as

$$G(x', z'; x_s) = \sum_j e^{ie_j \cdot (x'-x_s)} G(\xi_j, x, z; x_s) = \sum_j e^{i\kappa \cdot \mathbf{r}} G(\xi_j, x, z; x_s)$$  \hspace{1cm} (A8)

where $\mathbf{r}=(x'-x_0, z'-z)$, $x_0$ is the coordinate origin at the decomposition level, usually $x_0=0$, and $\mathbf{K}_j=(\xi_j, \xi_j)$ is the local wavenumber vector, with the local vertical wavenumber

$$\xi_j = \sqrt{(\omega/v)^2 - \xi_j^2}$$  \hspace{1cm} (A9)

where $v$ is local velocity.

Appendix C: Resolving kernel of double-focusing imaging process

Apply local 3D Fourier transform (local wavenumber-domain decomposition) on $\mathcal{R}(x, x_0)$ in (41) with coordinate center at $x_0$, resulting in the expression of PSF in local wavenumber domain

$$\mathcal{R}(\mathbf{K}, x_0) = -4\int_{A_f} d\omega k^2 \int d\mathbf{K} \int dx_s \frac{\partial G^*_f(\omega, x_0, \mathbf{K}; x_s)}{\partial z} G_M(\omega, x_0; x_s)$$  \hspace{1cm} (C1)

where $\mathbf{K}$ is the 3D local wavenumber vector and $\mathbf{K} = \mathbf{K} \hat{\theta}$, with $\hat{\theta}$ as the unit angular vector, $A_f$ is the frequency aperture (band), and the integration over $K_s$ is a convolution integral in $K$ domain and $K_s = k \hat{\theta}_s$, with $\hat{\theta}_s$ as a unit vector in the source direction. For a single frequency $\omega, \hat{k} = k = \omega/v(x_0)$ with $v(x_0)$ as the local velocity. We see also that $K_g = K_s = k \hat{\theta}_g$, with $\hat{\theta}_g$ as a unit vector in the receiver direction. In the above equation, $G^*_f(\omega, x_0, \mathbf{K}; x_s) G_M(\omega, x_0; x_s)$ can be considered as a point source at $x_0$ radiating wavefield to $x_s$, and then backpropagated (phase conjugate) again to $x_0$, with the local plane wave directions $\hat{\theta}_s$. If we use the high-frequency asymptotic approximation of Green’s function, then the incident wave direction will be determined by the ray direction. For the more general case of wave propagation, the local plane wave decomposition is more appropriate (Wu and Chen, 2002; Xie and Wu, 2002; Wu et al. 2004). In a similar way

$$\frac{\partial G^*_f(\omega, x_0, \mathbf{K}; x_s)}{\partial z} G_M(\omega, x_0; x_s)$$

corresponds to a point source at $x_0$, radiating wavefield to
\( x_g \), and then backpropagated to \( x_0 \), with the local plane wave direction \( \hat{\theta}_g \). The spectral component determined by this scattering experiment with a single frequency will be at
\[
\mathbf{K} = \mathbf{k}_g + \mathbf{k}_s = \mathbf{K}_k \hat{\theta}_k ,
\]
with
\[
\mathbf{K} = 2k \sin(\theta_{sc} / 2) ,
\]
and
\[
\theta_k = (\theta_g + \theta_s) / 2 , \quad \theta_r = \theta_{sc} / 2 = (\theta_g - \theta_s) / 2 ,
\]
where \( \hat{\theta}_k \) is the unit direction vector of \( \mathbf{K} \), \( k = \omega / c_0 \), with \( c_0 = v(x_0) \) and \( \theta_{sc} \) is the scattering angle defined as the angle between the incident direction and the receiving direction. \( \theta_{sc} \) is the angle of \( \mathbf{K} \) vector with respect to a reference direction, such as the z-axis; \( \theta_s \) and \( \theta_g \) are the angles of the \( \mathbf{k}_s \) and \( \mathbf{k}_g \) vectors respectively, and \( \theta_r \) is the reflection angle when dealing with local reflectors. For each frequency, we may find a pair of \( (\mathbf{k}_s, \mathbf{k}_g) \) satisfying \( \mathbf{K} = \mathbf{k}_s + \mathbf{k}_g \).

Therefore, equation (C1) can be written into another form:
\[
\mathfrak{Re}(\mathbf{K} = \mathbf{k}_s, \mathbf{x}_0) = -4 \int_{A_g} d\omega \mathcal{K}^2 \int_{A_s} dx_s \frac{\partial G_i^* (\omega, \mathbf{x}_0, \mathbf{k}_s; x_s)}{\partial z} G_M (\omega, \mathbf{x}_0, x_s)
\]
\[
\int_{A_g} dx_g \frac{\partial G_i^* (\omega, \mathbf{x}_0, \mathbf{k}_g; x_g)}{\partial z} G_M (\omega, x_g, \mathbf{x}_0)
\]

The final magnitude of the spectral component \( \mathfrak{Re}(\mathbf{K}, \mathbf{x}_0) \) is the sum of contributions from all the pairs. The effect of data aperture (domain of integration), including the frequency band and acquisition spatial aperture, is in the integration apertures \( A_f, A_s \) and \( A_g \).

References

Direct waveform inversion via iterative inverse propagation

R. Brian Schlottmann

Summary

Seismic waves are the most sensitive probe of the Earth’s interior we have. With the dense data sets available in exploration, images of subsurface structures can be obtained through processes such as migration. Unfortunately, relating these surface recordings to actual Earth properties is non-trivial. Tomographic techniques use only a small amount of the information contained in the full seismogram and result in relatively low resolution images. Other methods use a larger amount of the seismogram but are based on either linearization of the problem, an expensive statistical search over a limited range of models, or both. We present the development of a new approach to full waveform inversion, i.e., inversion which uses the complete seismogram. This new method, which falls under the general category of inverse scattering, is based on a highly non-linear Fredholm integral equation relating the Earth structure to itself and to the recorded seismograms. An iterative solution to this equation is proposed. The resulting algorithm is numerically intensive but is deterministic, i.e., random searches of model space are not required and no misfit function is needed. Impressive numerical results in 1D are shown for several test cases.

1 Introduction

Seismic waves provide the best and most direct probe available of the properties of the Earth’s interior. In exploration seismology artificially generated waves recorded at the Earth’s surface are commonly used to image material discontinuities in the subsurface (e.g. via migration). However, a more elusive goal is inversion, the determination of Earth properties such as wave velocity and density from seismic data. Tomographic inversions, which use only the traveltime information of isolated arrivals, are frequently used to obtain smooth, low resolution estimates of the wave velocity generally for use in imaging algorithms. Inversion methods which are designed to make use of all the information in the data, i.e., not only the arrival time and amplitude of pulses but the details of their shape as well, fall under the category of “waveform inversion”. The additional information contained in the waveforms promises to improve the accuracy and resolution of Earth structure models.

Although elegant in theory, waveform inversion has not enjoyed much use partly because it is computationally expensive—more than an order of magnitude more expensive than migration—and partly because it requires expert user intervention to make it work. However, in more
recent years, faster computers, better inversion strategies (e.g. Pratt, 1999a, 1999b; Sirgue and Pratt, 2004; Yokota and Matsushima, 2004), and a greater demand for accurate information on subsurface structure have made this approach more and more appealing.

Most of the work done to date in waveform inversion falls into one or both of two broad categories: A) linearized methods and B) model-searching methods. Model-searching methods (see e.g. Sen and Stoffa, 1995) involve three main components: 1) the definition of a misfit or error function that measures the difference between seismic waveform data and synthetic seismograms for a proposed Earth model; 2) a pre-defined set of adjustable model parameters and parameter ranges; and 3) some rule or set of rules for searching the defined parameter space. The general idea is to find the combination of model parameters that minimizes the error. This sort of approach can work well if the number of possible parameter combinations to be examined is small. Otherwise, given that the generation of synthetic seismograms is usually quite expensive computationally in three spatial dimensions, searching a large-dimensional model space is impractical at this time.

Linearized methods are, on the other hand, quite cheap computationally. A mainstay of this approach is to assume that the true, unknown Earth structure differs only slightly from some initial reference model. Under this assumption, one can assume that the true wavefield can be approximated by a combination of the wavefield in the reference model and a singly-scattered wavefield. This single-scattering assumption is known as the first-order Born approximation. Mathematically, this is written as

\[ \psi(x, \omega) \approx \psi_0(x, \omega) + \omega^2 \int d^3 \mathbf{x}' G_0(x, \omega; x') V(x') \psi_0(x', \omega), \]  

(1)

where \( \psi \) and \( \psi_0 \) are the actual and reference wavefields, respectively; \( G_0 \) is the Green's function of the reference medium; \( V \) is the difference between the reference and actual media (and is frequently known as the "scattering potential"); and \( \omega \) is the angular frequency. Because this approximation is linear in \( V \), one can construct algorithms to extract \( V \) and, hence, an approximation to the true model from measurements of \( \psi \) on the Earth's surface. Of course, there are many practical considerations which can make this process difficult to perform, not the least of which is insufficient data. However, even from a theoretical perspective, there are problems. Specifically, if the true structure differs more than a little from the reference model—the usual case—the data can contain arrivals, most notably multiples, that cannot be predicted from the single scattering approximation.

One approach to waveform inversion that does not fall into either the linearized or model-searching categories is what is known as inverse scattering. The name, "inverse scattering", comes from the knowledge that wavefields can be viewed as resulting from multiple scattering of wave energy within the Earth. The object is to find some way to invert these multiple scatterings to retrieve the Earth structure. The principal work done in this area uses an approach called the "inverse scattering series".

Originally developed by Jost and Kohn (1952) in quantum physics and later by Moses (1956), the properties of the inverse scattering series have been studied by Prosser (1980), among others.
Razavy (1975) was first to apply this method to the seismic inverse problem. In recent years, it has been most extensively explored by Weglein and collaborators (e.g. Weglein et al., 2003). The main idea of this approach is first to write the wavefield as measured on the Earth’s surface as a Born series, the multiple-scattering generalization of eq. (1). In a compact symbolic notation, this is written as

$$\psi_S = (\psi_0)_S + (G_0 V \psi_0)_S + (G_0 V G_0 V \psi_0)_S + \ldots,$$

where the subscript indicates that we evaluate the wavefield only where we can actually measure it. In principle all orders of scattering off the unknown structure \( V \) are included in this sum. If we also assume that the quantity \( V \) can be written as an infinite series,

$$V = V_1 + V_2 + V_3 + \ldots$$

where \( V_n \) is \( n \)-th order in the recorded data, we can insert this series into the Born series and collect terms of common order in the data. The result is an infinite set of equations for the \( V_n \):

$$\begin{align*}
(\psi - \psi_0)_S &= (G_0 V_1 \psi_0)_S \\
0 &= (G_0 V_2 \psi_0)_S + (G_0 V_1 G_0 V_1 \psi_0)_S \\
& \vdots
\end{align*}$$

These equations are meant to be solved in a cascade fashion, solving first for \( V_1 \) then using that result to obtain \( V_2 \) and so forth.

By studying these equations numerically in one spatial dimension, Carvalho (1992) has found that this approach does not appear to be convergent for an arbitrary contrast between the reference and the actual medium. Following his result, Weglein and collaborators have studied the sub-series approach, in which various terms in the full inverse scattering series of common form are combined into separate sub-series, each of which is postulated to perform a different task corresponding to a conventional processing step (Weglein et al., 2003). Using the sub-series approach, they have been able to achieve some promising results with no obvious problems with divergence.

We have derived an alternative approach to the problem of waveform inversion—one which also falls into the category of inverse scattering. We introduce this new development in the next section.

2 Theory

2.1 Derivation

To introduce and test the concepts of our approach, we present the theory in one spatial dimension. The data will consist of a single trace recorded at \( z = 0 \) for a source also located at \( z = 0 \). The source-time function will be a \( \delta \)-function, making this single trace the complete impulse
response for the chosen source/receiver pair. The unknown structure we will wish to recover is assumed to be located “beneath” the source/receiver, along the half-line \( z > 0 \). Further constraints we impose on the physics of the problem are that we have purely acoustic propagation, fully variable velocity, no attenuation, and constant density. This last constraint is imposed not only for simplicity but for the reason that it is not possible to reconstruct both velocity and density from a single trace in 1D.

We take as our reference, or background, a constant-velocity medium with wave propagation velocity \( c_0 \) and associated Green’s function \( G_0 \). Let \( c(z) \) and \( G \) be the wave velocity and Green’s function of the true medium, respectively. We define the scattering potential as

\[
V(z) = \frac{c^2}{c_0^2} - 1.
\]  

(5)

The basis of our derivation will be what is known as the Lippmann-Schwinger equation, the integral equation equivalent to the more frequently seen differential equation for acoustic wave propagation:

\[
G(z, \omega; z_0) = G_0(z, \omega; z_0) + \frac{\omega^2}{c_0^2} \int_{-\infty}^{\infty} dz' G_0(z, \omega; z') V(z') G(z', \omega; z_0).
\]  

(6)

One method of solving this equation results in the generation of the Born series or, at least, the generation of Born approximations of various orders. For instance, the first-order Born approximation (also known simply as the Born approximation) is obtained by approximating \( G \) by \( G_0 \) above on the right only, yielding

\[
G(z, \omega; z_0) \approx G_0(z, \omega; z_0) + \frac{\omega^2}{c_0^2} \int_{-\infty}^{\infty} dz' G_0(z, \omega; z') V(z') G_0(z', \omega; z_0).
\]  

(7)

The second-order Born approximation can be obtained by first replacing \( G \) on the right-hand side of eq. (6) with the entirety of the right-hand side,

\[
G(z, \omega; z_0) = G_0(z, \omega; z_0) + \frac{\omega^2}{c_0^2} \int_{-\infty}^{\infty} dz' G_0(z, \omega; z') V(z') G_0(z', \omega; z_0)
+ \frac{\omega^4}{c_0^4} \int_{-\infty}^{\infty} dz' \int_{-\infty}^{\infty} dz'' G_0(z, \omega; z') V(z') G_0(z', \omega; z'') V(z'') G(z'', \omega; z_0),
\]  

(8)

and then setting \( G \approx G_0 \) on the right again. The general rules for obtaining approximations of arbitrary order are

\[
G^{(1)}(z, \omega; z_0) = G_0(z, \omega; z_0)
\]
\[
G^{(n)}(z, \omega; z_0) = G_0(z, \omega; z_0) + \frac{\omega^2}{c_0^2} \int_{-\infty}^{\infty} dz' G_0(z, \omega; z') V(z') G^{(n-1)}(z', \omega; z_0),
\]  

(9)

where \( G^{(n)} \) indicates the \( n \)-th approximant to \( G \). For our purposes, however, the once-iterated equation without the substitution of \( G_0 \) for \( G \), eq. (8), is what we want.
To greatly shorten some of the equations to follow, we define some helpful notation. Setting $z = z_0 = 0$, we let

$$
(G_0 V G_0)_{RS} = \frac{\omega^2}{c_0^2} \int_{-\infty}^{\infty} dz' G_0(0, \omega; z') V(z') G_0(z', \omega; 0)
$$

(10)

and

$$
(G_0 V G_0 V G)_{RS} = \frac{\omega^4}{c_0^4} \int_{-\infty}^{\infty} dz' \int_{-\infty}^{\infty} dz'' G_0(0, \omega; z') V(z') G_0(z', \omega; z'') V(z'') G(z'', \omega; 0),
$$

(11)

where the subscript $RS$ indicates that the quantities are evaluated only for the given source/receiver geometry. Thus, our once-iterated equations becomes

$$
G(0, \omega; 0) = G_0(0, \omega; 0) + (G_0 V G_0)_{RS} + (G_0 V G_0 V G)_{RS}.
$$

(12)

Using the explicit form of the Green’s function $G_0$,

$$
G_0(z, \omega; z_0) = \frac{i c_0}{2 \omega} e^{i \omega|z-z_0|/c_0},
$$

(13)

and the fact that $V(z) \equiv 0$ for all $z < 0$, we find

$$
(G_0 V G_0)_{RS} = \frac{\omega^2}{c_0^2} \int_0^{\infty} dz \left( \frac{i c_0}{2 \omega} \right)^2 V(z) e^{2i\omega|z|/c_0}
$$

$$
= -\frac{1}{4} \sqrt{2\pi} \tilde{V}(2\omega/c_0),
$$

(14)

where $\tilde{V}$ is the Fourier transform of $V$. Inverting this relationship, we find

$$
V(z) = -\frac{4}{\pi c_0} \int_{-\infty}^{\infty} d\omega e^{-2i\omega z/c_0} (G_0 V G_0)_{RS}.
$$

(15)

(Note that our Fourier transform convention is that of symmetric normalization for the inverse and forward transforms.) Letting $\tilde{D}(\omega) = G(0, \omega; 0) - G_0(0, \omega; 0)$ and

$$
U(z) = -\frac{4}{\pi c_0} \int_{-\infty}^{\infty} d\omega e^{-2i\omega z/c_0} \tilde{D}(\omega),
$$

(16)

which is tantamount to a constant-velocity depth migration of the data, we apply the above inverse Fourier transform to the once-iterated equation and rearrange to get

$$
V(z) = U(z) + \frac{4}{\pi c_0} \int_{-\infty}^{\infty} d\omega e^{-2i\omega z/c_0} (G_0 V G_0 V G)_{RS}.
$$

(17)

Noting that $G$ on the right-hand side depends on $V$, we see that this equation is an inhomogeneous, highly non-linear Fredholm integral equation of the second kind for the scattering potential $V$. This equation is a novel result and forms the basis for our approach to waveform inversion.
2.2 An iterative method of solution

There is no pre-existing mathematical apparatus for solving equations like eq. (17). One possible scheme for solving it—one that should eventually be vetted by mathematicians—can be obtained by analogy to the iterative way of solving the Lippmann-Schwinger equation that we discussed earlier (eqs. 9). We propose to construct a sequence of approximants to $V$ by the following rules:

\begin{equation}
V^{(0)}(z) = 0
\end{equation}

\begin{equation}
V^{(n)}(z) = U(z) + \frac{4}{\pi c_0} \int_{-\infty}^{\infty} d\omega e^{-2i\omega z/c_0} \left( G_0 V^{(n-1)} G_0 V^{(n-1)} G^{(n-1)} \right)_{RS}
\end{equation}

where we generalize the notation of eq. (11) by substituting $V^{(n-1)}$ for $V$ and use $G^{(n-1)}$ to indicate the Green’s function for a medium with potential $V^{(n-1)}(z)$. Going back to the once-iterated Lippmann-Schwinger equation, eq. (12), which holds for any medium, we have

\begin{equation}
\left( G_0 V^{(j)} G_0 V^{(j)} G^{(j)} \right)_{RS} = G^{(j)}(0,\omega;0) - G_0(0,\omega;0) - \left( G_0 V^{(j)} G_0 \right)_{RS}.
\end{equation}

Using this relation in our iterative equation, we can achieve a substantial simplification to our second iteration rule above:

\begin{equation}
V^{(n)}(z) = U(z) + V^{(n-1)}(z) + \frac{4}{\pi c_0} \int_{-\infty}^{\infty} d\omega e^{-2i\omega z/c_0} \left( G^{(n-1)}(0,\omega;0) - G_0(0,\omega;0) \right).
\end{equation}

In parallel with our definition of $U$, eq. (16), we set

\begin{equation}
U^{(n-1)}(z) = -\frac{4}{\pi c_0} \int_{-\infty}^{\infty} d\omega e^{-2i\omega z/c_0} \left( G^{(n-1)}(0,\omega;0) - G_0(0,\omega;0) \right)
\end{equation}

to get the final version of our iteration rules:

\begin{equation}
\begin{align*}
V^{(0)}(z) &= 0 \\
V^{(n)}(z) &= U(z) + V^{(n-1)}(z) - U^{(n-1)}(z).
\end{align*}
\end{equation}

The algorithmic interpretation of these iteration rules is simple. At each step, the new approximant $V^{(n)}$ is obtained from the previous one, the “migrated” data $U$, and the migrated synthetic data from a medium with scattering potential $V^{(n-1)}$. Thus our algorithm should recover $V$ through a deterministic sequence of forward modelling simulations. It is for this reason that we call our approach to inverse scattering “iterative inverse propagation”.

3 Numerical Examples

As proof of concept, we present three synthetic tests of 1D iterative inverse propagation. In each case, we constructed a test model and generated synthetic seismograms with a staggered-grid
finite-difference (FD) code (see e.g. Virieux, 1986) using a Gaussian source wavelet with a half-width of about 0.5 s. In the inversion, we used an initial constant background velocity of 1 km/s, and the necessary forward modelling was done with the same FD code and source wavelet that was used to generate the synthetic data. We emphasize that no pre-processing, such as removal of multiples, was performed on the data.

Figure 1: One layer of thickness 3 km with a wave velocity 40% above background. Comparison of the true model (in red) and the inversion results (in black) are shown for various iterations indicated in each panel by the parameter N.

For numerical reasons, one deviation from the derived iteration rules, eq. (23), was used. At each iteration, the model obtained from the previous step was used as the new “mapping velocity”, i.e., the velocity used to map time into space. In other words, instead of using the factor of $2z/c_0$ in the exponential of eq. (23), a more general function $T(z)$—the travelt ime from the source to the point $z$ and back to the receiver—was computed from the previous iteration. This very practical 	extit{ad hoc} feature was used to eliminate some instabilities that were otherwise occurring at the bottom end of the structure of any test attempted. In fact, this approach also accelerated convergence and indicates the need to incorporate non-constant reference models in future work.
Figure 2: A smoothed random model with velocity variations of approximately ±40% about background. Note that the inversion recovers even the gradient at the end of the model.

Figure 1 shows the results for our first test, a model with a single high-velocity layer imbedded in an otherwise homogeneous medium. The results of the first three iterations and iterations 10, 20, 30, and 40 are shown. Both the amplitude of the velocity jump (40% above reference) and the positions of the top and bottom of the layer are quite well recovered, with the essential features of the true model recovered by iteration 20 and only moderate improvement obtained thereafter. Those deviations from the true model that are present are due to the band-limited wavelet used in both the initial synthetic data and the FD modelling performed in the inversion.

Figure 2 demonstrates the ability of the algorithm to reconstruct the details of a complicated, smooth model with fluctuations of approximately ±40% about the reference velocity. We see that by iteration 10 the errors in the inversion result are negligible. It is particularly notable that even the smooth gradient at the end of the model is recovered.

Finally, Figure 3 shows the results for a single low-velocity layer of velocity 40% below reference. The value of this example is that it shows clearly what happens to multiples as the inversion progresses. Specifically, one sees that they are gradually “pushed” upward into the
Figure 3: One low-velocity layer of thickness 3 km with a wave velocity 40% below background. Note the progressive collapse of multiples into the main structure.

main structure, indicating that the information they provide is instrumental in obtaining an accurate final result.

4 Discussion

We have shown the potential of iterative inverse propagation to recover the details of wave velocity structures in 1D. There are, of course, going to be many hurdles to overcome in extending this method to practical use. In addition to incorporating non-constant reference media, we must also accommodate irregular source/receiver geometries, elasticity, attenuation, limited frequency ranges, and unknown source wavelets, just to name a few. There will also be the issue of the enormous computational effort necessary to perform the forward wave propagation computations required by the algorithm.
However, it is likely that all of these will be overcome in time, and the straightforward nature of a deterministic algorithm such as this implies substantial long-term benefits to seismic exploration.

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References


Local angle information extraction and its application to velocity updating

Hui Yang, Xiao-Bi Xie and Ru-Shan Wu

Summary

The pre-stack wave-equation migration has been shown as a powerful tool for migration velocity estimation. The local angle image gathers provide much information which allows us to conduct the illumination analysis, AVA analysis, and amplitude correction, etc. In addition, the angle imaging gather also provide the basis for migration velocity updating. In this paper, we propose to use the information from the local image matrix and the local tau-matrix, both of them are in the local angle domain, for migration velocity updating. The structure dipping angle is considered in the new method and the true reflection angles related to the structure is used. We first describe how to extract the image amplitude and local time delay information in local angle domain. Then we demonstrate their applications in interval velocity updating by migration residual moveout analysis and travel-time inversion using the local time delay. As a numerical example, we use a constant layered velocity model with curved interfaces to demonstrate the feasibility of this approach, and show some preliminary results.

Introduction

An accurate velocity model is the key to the correct imaging of complex structures. Migration velocity analysis (MVA) method is a powerful technique for velocity updating. This type of velocity analysis basically uses the flatness of events in common image gathers (CIGs), traditionally sampled in offset domain, as a criterion for estimation of velocity quality (Al-Yahya, 1989). Comparing to the offset domain CIGs, the angle domain common image gathers generated by wave-equation based pre-stack migration methods have their inherent advantages to handle reflector ambiguity due to the multipathing in complex velocity models (Nolan and Symes, 1996; Xu et al., 1998; Brandsberg-Dahl et al., 1999; Prucha et al, 1999). Consequently, many literatures presented the migration velocity analysis methods based on angle domain common image gathers (e.g., Mosher, Jin and Foster, 2001; Biondi, et al. 2004).

However, most of these approaches are based on assumptions of small dip, small offset, and/or constant velocity layered models (e.g., Al-Yahya, 1989; Liu and Bleistein, 1992; Meng et al., 1999; Jiao, et al., 2002; Sava, et al., 2003), or the effects from local structure dipping angles were neglected without being put as parameters for inversion, leading to systematic errors in velocity updating. Only Biondi’s (2004) approach, based on the locally constant velocity assumption, expresses the migration residual moveout as the function of apparent structure dipping angle and reflection angle.

In this paper, we will decompose the downward extrapolated source and receiver side wavefields in pre-stack wave-equation migration process to construct the local imaging matrix (Wu and Chen, 2002; Xie and Wu., 2002; Wu and Chen, 2003) and local tau-matrix (Wu, 2006). The local imaging matrix is composed of angle domain partial images from all scattering events. The information on reflector dipping angle, the true angle domain common image gathers related to
the dipping angle and the local time delay can be obtained from these local matrixes. We then propose to use the amplitude and phase information in these matrixes for velocity updating.

In the following sections, we first describe how to extract the local structure dipping angle, the common imagine gather in true reflection angle domain, and local time delay from the downward extrapolated wavefields. Then, we will show how these angle information can be used for the velocity updating. The image depth residual moveout can be used for conventional velocity scanning and the angle domain delay time can be used for velocity updating. At last, a two-dimensional velocity model is used as a numerical example to demonstrate the application of the new velocity updating method.

**Review of Local Image Matrix**

The local image matrix (LIM) is a function of the incidence-scattering angle pairs at the target region and can be seen as an expansion of the conventional image condition. It can be defined as

\[
L_u(\theta_i, \theta_g, x, z, \omega) = \int \int W_s(\theta_i, x, z, x_s, \omega) U^*_s(\theta_g, x, z, x_g, \omega) dx_s dx
\]

(1)

where \( \omega \) is the frequency, \( \theta_i \) is the local incidence angle, \( \theta_g \) is the local receiving angle (see Fig 1), \( W_s(\theta_i, x, z, x_s, \omega) \) is the local incidence plane wave (incident beamlet) from a source at \( x_s \), \( U^*_s(\theta_g, x, z, x_g, \omega) \) is the local scattered plane wave (scattered beamlet) to the receiver at \( x_g \), and \( A_s \) and \( A_g \) are the source and receiver apertures, respectively. The decomposition of the wavefield can be conducted within a space window of size \( a \), and using either the GDF (Gabor-Daubechies
Frame) transform (Wu and Chen, 2002; Wu and Chen, 2003), windowed Fourier transform or a local slant stacking (local Radon transform) (Xie and Wu, 2002). Different types of common image gathers, e.g., common reflection-angle image gather and common dip-angle image gather, can be extracted from local image matrix (Xie and Wu, 2002; Wu and Chen, 2002; Wu and Chen, 2003).

**Definition of Local Tau-Matrix**

Usually the phase (travel-time) information is more reliable than the amplitudes, and travel-time can be directly correlated to the interval velocities. We discuss how to use the phase mismatch information to estimate or update the interval velocities. At the specific interface of the reference model (either from the starting model or from the result of previous iterations), phase (or travel time) errors in local angle domain can be obtained from the local image matrix. We name this as the “local τ matrix” or “local travel-time error matrix” which can be calculated using the Fourier transform with proper phase unwrapping (Wu, 2006)

\[ L(\theta_x, \theta_y, x, z, \omega) = \int_{\Delta \omega}^{\omega_{\Delta \omega}} L(k_x, k_y, x, z, \omega) \exp[-i\omega(t - t_0)] d\omega. \]  

(2)

Then the complex local image matrix is

\[ L(k_x, k_y, x, z, \omega) = A_L \exp(i\tau_L), \]  

(3)

where \( A_L = A_L(k_x, k_y, x, \omega) \) is the amplitude of matrix element and \( \Phi_L = \Phi_L(k_x, k_y, x, z) \) is the phase of matrix element and

\[ \tau_L = \omega \Phi(k_x, k_y, x, z) = \tau(k_x, k_y, x, z) \]  

(4)

is local tau-matrix.

After obtained the time delay \( \tau_L \) in the local angle domain, we can conduct an inversion by back-project the travel time error \( \tau(k_x, k_y, x, z) \) to the velocity model. For simple cases such as smoothed velocity model, a ray back-projection is the simplest updating method. In this case, \( \tau(k_x, k_y, x, z) \) will be back-projected to the round trip path as caused by velocity errors along the path with the following relationship,

\[ \frac{r_m(\theta_x, x, z)}{v_m} - \tau(\theta_x, x, z) = \frac{r_i(\theta_x, x, z)}{v_i}, \]  

(5)

where \( \tau(\theta_x, x, z) \) is the travel time error sampled at the true reflection angle \( \theta_x \) and the scattering point \((x, z)\), \( v_m \) is the erroneous migration velocity and \( v_i \) is the true velocity, \( r_m(\theta_x, x, z) \) and \( r_i(\theta_x, x, z) \) are lengths of round trip ray paths in the layer above the scattering point for true velocity and erroneous migration velocity, respectively. For small velocity errors, we assume that \( r_i(\theta_x, x, z) \) approaching \( r_m(\theta_x, x, z) \) and the updated velocity \( v_i \) can be calculated. To update the entire velocity model, iteration and the layer stripping techniques can be adopted.

**Local Angle Information Extraction and Residual Velocity Analysis**
To demonstrate the extraction of local angle information, we build a four-layer model with constant velocity in each layer (shown in Fig 2). The interfaces of Baina, et al. (2002) are used in the model and the velocities in these layers are 3.5 km/s, 3.7 km/s, 4.0 km/s and 4.2 km/s, respectively. The lateral and vertical sampling intervals are 25 m and 10 m respectively. We generate 201 shot gathers with a second-order full-wave acoustic finite-difference code. The shot interval is 50 m, and the largest offset is 7 km with a split-spread geometry. A Richer wavelet is used as the source time function and the dominant frequency is 17.5 Hz. The time sampling interval is 8 ms and the total recording time is 3.5 sec. We use the local cosine basis beamlet pre-stack wave-equation migration method (Luo, et.al., 2004) for imaging and extract the local angle information from the migrated wavefield.

In order to test the velocity updating method, we keep the depths of the interfaces unchanged but use the following velocities as the starting model in the migration calculation: 3.8 km/s, 3.9 km/s, 4.0 km/s and 4.1 km/s. Shown in Fig 3 is the image using erroneous velocities compared to the true interfaces (shown in red lines). The image is obviously distorted and artifacts are emerged at some locations.

Fig 4 shows the process of extracting angle related information from the migrated wavefield. Fig 4a is the migrated image using the erroneous velocity model where the column labeled CIP186 is the location we will conduct the measurement. Shown in Fig 4b are two local image matrices at two specific points indicated by arrows. Fig 4c shows the common image gather sorted by reflection angles with the red solid lines indicating the theoretical residual moveout based on flat layer and constant velocity assumption. Shown in Figs 4d and 4e are dipping angle gather and dip scan gather at CIP186. The scan gather is obtained by calculating the cross-correlation of traces with respect to a reference trace (obtained by laterally average all traces). The apparent dipping angles picked from the dip scan gather are more accurate than directly from the dipping angle gather. As a comparison, the true dip angle of the third reflector is indicated by a blue dash line and the red dash line indicates the zero dipping angle. Due to the erroneous velocity model and large curvature of the interface, the first reflector is not showing very clear.

Based on the common image gather and local structure dipping angle, residual velocity scan (Yang, et al., WTOPI Technique Report Vol. 12, 2005) is performed. Fig 5 shows the velocity scan result at distance 2 km, where (a) is the common image gather sorted by true reflection angle, (b) is the residual scan profile. The solid line in panel (b) indicates the true velocity. The energy peaks and true velocity are consistent for simple reflector structures. Similarly, Fig 6 shows the result at distance 6 km, where (a) is the common image gather and (b) is the corresponding velocity scan profile. The solid line on panel (b) denotes the true velocity. There are apparently errors due to the large dipping angle and strong curved interfaces, as can be seen especially from the third peak in Fig 6(b).

**Velocity Analysis Based on the Local Tau-Information**

We see from the previous section that the migration residual velocity scan including the local structure dipping angle can be used to estimate the velocity for simple structures. Nevertheless, its result is not satisfied within complex regions with curved interfaces or lateral velocity variations due to over simplified assumptions adopted.
To make the velocity updating more robust in models with large dipping angles and curved interfaces, we include the local time delay information in the velocity analysis. Fig 7 shows the $\tau(\theta)$ gathers for erroneous migration velocity and true migration velocity. The investigating point is at distance 2 km (see Fig 4(a)). The true depth of the first reflector is 1.96 km, i.e. point (2, 1.96). There is a 10 percent higher velocity in first layer. (a) shows the curved-up $\tau(\theta)$ gather of point (2, 1.96) at true layer interface. (b) shows $\tau(\theta)$ gather at point (2, 2.16), the imaging point of the erroneous migration velocity. Events near zero reflection angles approach to zero time delay, but the time delay remains at large reflection angle because of the wrong migration velocity. As a comparison, (c) shows $\tau(\theta)$ gather for true velocity at point (2, 1.96), which is on the first true interface. The events align at zero time indicating the correct imaging point and migration velocity.

Fig 2: Constant layer velocity model with curved interfaces

Fig 3: The depth image calculated from erroneous layer velocity model. The red thick lines indicate the location of true interfaces.
Local Angle Information and Velocity Updating

Fig 4: The process of extracting angle related information from the migrated wavefield. (a) Migration image using the erroneous velocity model. (b) The two local image matrices calculated for specific locations indicated by arrows. (c) The reflection angle gather with theoretical residual move-out (red lines). (d) and (e) Dipping angle gather and dip scan gather for CIP186 labeled in (a) (details see text).
Fig 5: Common image gather (a) and residual velocity scan profile (b) at 2 km point. The picked velocity is very close to true velocity (yellow line on velocity scan profile).

Fig 6: Common image gather (a) and residual velocity scan profile (b) at 6 km point. The picked velocities have apparent errors comparing to the true velocities especially at the third peak (yellow line on velocity scan profile).

To show the application of the local tau-information on the numerical example, the local time delay at the interface point (the first reflector) was extracted. The migration velocity is 6 percent lower than true velocity in the first layer. Fig 8 shows the $\tau(\theta)$ gather deduced from local tau-matrix with the true dip angle information at the specific point on first reflector. The
investigating point is at distance 2 km. Fig 9 shows the \( \tau(\theta) \) gather at 5.45 km on first reflector. Obviously there are significant time delays for different reflection angle. The time delay for each reflection angle is picked to evaluate the velocity error, and ray based back-projection is applied to update to the related model.

Shown in Fig 10 is the result of velocity updating for the upper three layers in the model. The green line denotes the initial velocity model used in migration. The red dash line denotes the velocities estimated using the information from the tau-matrix. The blue solid line denotes true velocities. The velocities obtained using the local tau-matrix is very close to true velocities.

**Conclusions**

Local image matrix can be used to provide many angle related information, such as the dip angle of the reflector, local time delay, and this information can be further applied to residual migration velocity analysis and travel time tomography. We demonstrated how to extract this information from the migrated wave-field and how to use this information in migration velocity analysis. Results from this paper are preliminary since some a priori information is used in the inversion. Further investigations are required to extend the method to the more general cases.

![Figure 7](image)

**Figure 7:** \( \tau(\theta) \) gather at distance 2km. The true depth of the first reflector is 1.96 km, i.e. point (2, 1.96). There is a 10 percent higher velocity in first layer. (a) \( \tau(\theta) \) gather of point (2, 1.96) at true layer interface. (b) \( \tau(\theta) \) gather at point (2, 2.16), the imaging point of the wrong velocity. (c) \( \tau(\theta) \) gather at point (2, 1.96) at true layer interface for true velocity. The events from different reflection angles align at zero time.
Fig 8: $r(\theta, \tau)$ gather at the first reflector, the investigate point at distance 2km with 6 percent lower velocity in first layer.

Fig 9: $r(\theta, \tau)$ gather at the first reflector, the investigate point at distance 5.45km with 6 percent lower velocity in first layer.
Fig 10: The estimated velocities in upper three layers are very close to true velocity. Red dash line denotes the estimated velocity. Blue solid line denotes true velocity.

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PART III

NOVEL PROPAGATORS AND APPLICATIONS
A super-wide angle one-way propagator and imaging using turning waves

Ru-Shan Wu and Xiaofeng Jia

Summary

To overcome the angle limitation of regular one-way wave propagation methods, we propose to extend the capability of one-way propagators by a wavefront reconstruction method which combines and interpolates the two orthogonally propagated one-way wavefields. The proposed method has accurate super-wide angle (greater than 90º) propagation and can model turning waves. The method can be used in imaging steep subsalt reflectors and overhanging salt flanks. Numerical tests demonstrated the validity of the method in modeling and in imaging.

Introduction

Although significant progress has been made in improving the wide-angle accuracy of one-way wave propagation methods (Collins and Westwood, 1991; Ristow and Ruhl, 1994; Wu, 1994, 1996; Wu and Xie, 1994; Xie and Wu, 1998; Grimbergen et al., 1998; Jin and Wu, 1999a, b; Jin et al., 1999, 2000; Huang et al., 1999a, b; Xie et al., 2000; De Hoop et al., 2000; Le Rousseau et al., 2000; Han and Wu, 2005; Thomson, 2005), the accuracy of wide-angle waves for strong contrast media is still a serious problem and put a practical limitation on applying these methods to steep reflector imaging, especially in complex subsalt regions. The other more fundamental limitation of the one-way methods is the difficulty in handling turning waves and therefore renders the method of little use in imaging overhanging salt flanks.

In this work we propose to extend the capability of one-way propagators by a wavefront reconstruction method which combines and interpolates the two orthogonally propagated one-way wavefields to rebuild the distorted wavefront to good accuracy. The reconstruction will be conducted iteratively and the wavefront expansion can be accurately simulated well beyond 270º (135º single-side). Numerical examples of impulse response in c(z) medium and two-layered medium demonstrate the good accuracy of super-wide angle waves and the modeling of turning waves. The efficiency of the algorithm and potential of applications are discussed. An example of salt boundary imaging using synthetic data sets demonstrates the feasibility of imaging overhanging flanks by turning waves using this propagator.

Reconstruction of Accurate Wavefront Using Two Orthogonally Propagated One-Way Waves

One-way wave equation has a preferred direction for wave evolution (propagation). In this paper we focus only on the two-dimensional case. In Cartesian coordinate system, the preferred direction is either the z-axis or a horizontal direction in the 2D case. In exploration geophysics, normally z-axis is the preferred direction. In such a case, the large-angle waves, e.g. the waves with propagating angles exceeding 70º, inevitably carry some errors both in phase and amplitude. However, large-angle waves with respect to z-axis become small-angle waves to x-axis.
Therefore, we propose to use a wavefront reconstruction method combing two orthogonally propagated waves for good accuracy in super-wide angle ranges.

Assume two one-way equations in x and z directions:
\[
\frac{\partial u(x, z)}{\partial z} = P_z u(x, z) \\
\frac{\partial u(x, z)}{\partial x} = P_x u(x, z)
\]
where \( u(x, z) \) is the wavefield, \( P_z \) and \( P_x \) are the one-way propagators in z- and x-directions respectively. The formal solution for one step forward propagation can be written as
\[
u(x, z + \Delta z) = P_z[u(x, z)] \\
u(x + \Delta x, z) = P_x[u(x, z)]
\]
where \( \Delta x \) and \( \Delta z \) were incorporated into the propagators in the right-hand of equation (2). Now we consider the one-way propagation in x-direction having M steps forward, and in z-direction having N steps forward with respect to a reference point \( x_0 = (x_0, z_0) \):
\[
u(x, z + n\Delta z) = P_z^n[u(x_0, z_0)] \quad n = 1, \cdots N \\
u(x + m\Delta x, z) = P_x^m[u(x_0, z_0)] \quad m = 1, \cdots M
\]
We reconstruct the accurate wavefield by taking a weighted average of the two one-way wavefields. The wavefield reconstructed will be recorded as the accurate wavefield for modeling or imaging. The wavefront reconstructed will be used for further one-way propagation (see Figure 1). We know that the one-way propagation is exact in the forward direction and very accurate for small-angle waves, especially for good one-way propagators. Therefore, at any location we put heavy weight on the one-way solution which propagates small-angle waves at that point and light weight on the other one-way solution which propagates large-angle waves. This is to say that we set up a weighting scheme according to the local propagating angles:
\[
u(x, z + n\Delta z) = P_z^n[u(x_0, z_0)] + P_x^m[u(x_0, z_0)] \\
u(x + m\Delta x, z) = P_x^m[u(x_0, z_0)] + P_z^n[u(x_0, z_0)]
\]
where \( \theta_x \) is the propagating angle with respect to the x-axis, and \( \theta_z \), to the z-axis; \( w_x(\theta_x) \) and \( w_z(\theta_z) \) are the weight functions for the one-way fields along z-axis and along z-axis, respectively. M and N determine how often wavefront reconstruction is conducted. N=M=1 is the case of stepwise continuous reconstruction; on the other hand, N=N_z, M=M_x, corresponds to a simple scheme of weighted average without updating the wavefront, where \( N_z \) and \( M_x \) are the total sampling numbers of the model in z- and x-directions. Updating the wavefronts are important for modeling turning waves, refracted and diffracted waves in strongly heterogeneous media.

The weighted summation for reconstruction can be done either in space domain or in wavenumber domain. More efficiently it should be performed in local angle domain (beamlet domain) using beamlet propagators (e.g., Wu et al., 2000). In the local space domain, the propagating angles are determined by the local updating geometry. In the local angle domain, the propagating angle has a direct relation with the local wavenumber. We preferred to adopt a weighting scheme in local angle domain. There is an additional advantage of doing weighting in angle domain. Since the weighting in effect is a high-angle filtering, it is much more stable than the weighting in space domain. The weighting summation can be also applied to other one-way...
propagators, such as the GSP (generalized screen propagators) (Wu, 1994, 1996; Jin et al., 1998, 2002; Huang et al., 1999a, b; Xie and Wu, 1998). In this work, we test the scheme using the wide-angle Padé GSP algorithm (Xie and Wu, 1998). Different weight functions have been tested (see Figure 2). We found the Hanning type weighting gives the best results. In this scheme, the field with propagating angle less than 22.5º will be kept untouched, and that greater than 67.5º will be discarded. The field with mid-angles will be weighted accordingly.

Figure 1: wavefield and wavefront reconstructions by taking a weighted average of two one-way wavefields

Figure 2: weight functions used for wavefield reconstruction
$\theta$ is the propagating angle which is the angle between the propagating direction and the preferred direction. $\theta_c$ is the cut angle of the Hanning window.

**Impulse Responses in Different Media**

In this section several numerical examples will be shown using the method mentioned above and the comparison with the FD method will be discussed. For simplicity, in this section we employ $N=M=N_x=N_z$ in equation (4) to see the effect of simple weighted sum of two orthogonally propagated GSP one-way fields. The first model in Figure 3(a) is a $c(z)$ model with moderate gradient of velocity variation, i.e. $c=c_0+|z-z_s|c_0/h_1$. For all the models mentioned below, $c_0=2\text{km/s}$, $h_1=2.55\text{km}$ and $x_s=z_s=2.55\text{km}$. The source is a single point source and the dominant frequency of the wavelet is 30Hz. The modeling results are given as the snapshots for $t=0.6$, 0.75, 0.9, 1.05 and 1.2s. We see that the one-way modeling with wavefront reconstruction can simulate wide-angle propagating waves quite accurately. Due to the help of one-way propagation in the lateral
direction (x-direction), excellent accuracy can be achieved for propagating angles well beyond 90°. In the next section we will see that this feature extends the one-way method to a new area of modeling turning waves.

![Diagram](image)

Figure 3: Impulse response for a c(z) model
(a) Velocity model; (b) FD modeling; (c) modeling using one-way propagators with wavefront reconstruction

![Diagram](image)

Figure 4: modeling results for a two-layered model with a single point source
(a) Velocity model; (b) FD modeling; (c) modeling using one-way propagators with wavefront reconstruction

The model for Figure 4 is a two-layered model with 50% velocity contrast. The depth of the interface is $z=h_1-h_2$ with $h_2=3h_1/4$. Except the reflected waves, the wide-angle direct waves and transmitted waves are all well modeled.

In our experiments, we also tested line sources (source arrays) propagating in c(z) model. The source array is composed of 100 single point sources, which covers about one-fifth of the surface. The modeling results are shown in Figure 5. The model is the same as that for Figure 3.
Figure 5: modeling results for c(z) model with a source array
(a) Velocity model; (b) FD modeling; (c) modeling using one-way propagators with wavefront reconstruction

Figure 6: wavefront improvement by the wavefront reconstruction method
(a) Impulse responses for the regular GSP method and the new super-wide angle propagator in a c(x,z) medium; (b) maximum amplitudes picked from the wavefields generated by the two methods (blue: regular one-way method; red: the new propagator) compared with that of the finite difference method (green color).

The wavefront reconstruction method has a great advantage over the regular GSP method when the propagation angle is extremely large. This is shown in Figure 6 clearly. The model is c(x,z) model in which \( c = c_0 + |x-x_s|c_0/h_1 + |z-z_s|c_0/h_1 \). The time of snapshot is 0.9s. Since the regular GSP method has limited accuracy for waves with very wide angles, the wavefront has been...
distorted significantly at these wide angles (inner wavefront in Figure 6a). However, the new propagator reconstructs the wavefront very well (outer wavefront in Figure 6a). Figure 6b shows the positions of the maximum amplitudes. We see that the wavefront generated by the new propagator (red cooler) is almost the same as that for the FD method (green color); while the regular one-way propagator has significant distortions (blue color).

\[ c = c_0 \]

\[ h_1 = \frac{c_0}{c_0 + 2|z - z_s| c_0 / h_1} \]

(a) (b)

(c)

Figure 7: modeling turning waves by the new propagator
(a) velocity model; (b) snapshots obtained by the wavefield reconstruction method; (c) synthetic seismogram observed on the surface. The horizon coordinate is reduced traveltime and the vertical is trace number on the surface (the point source is in the middle). The direct arrivals are weakened to render the signals of turning waves more clearly.

We now discuss the computational efficiency of the methods compared with the regular GSP or other one-way methods. If we use the global propagator such as the GSP or FFD methods, the weighted one-way method with wavefront reconstruction will take roughly twice the computation time as the regular one-way methods. However, if we use the localized propagators such as the beamlet propagators (Wu et al., 2000), the efficiency of the wavefront reconstruction
method should increase, since the wavefront reconstruction near the source region do not need a full length calculation in the lateral direction. Compared with the full-wave finite difference method, the computation efficiency of the proposed method is still advantageous.

**Turning Wave Simulation**

The wavefield reconstruction method using the weighted average of two one-way wavefields can be used to model turning waves. We designed a $c(z)$ model with large velocity gradient to test the performance of the method for turning waves. Figure 7 shows the model (Figure 7a) snapshots (Figure 7b) and the surface records (Figure 7c). As shown in Figure 7(c), turning waves can be seen clearly for the far offsets and the traveltimes of the turning waves become less than those of the direct arrivals. In contrast, for the regular one-way method with preferred direction along the z-axis, turning wave modeling is a great obstacle.

![Figure 8: Imaging salt dome by the new super-wide angle one-way propagator](image)

(a) Velocity model; (b) surface records obtained by the FD method; (c) image obtained using the new propagator

**Imaging Salt Dome Flanks by the New One-Way Propagator**

As we have mentioned above, the new propagator can model the turning waves accurately than the regular one-way method. In some cases it can model the turning waves that are impossible to calculate with regular one-way propagators. This advantage indicates that the new method is very promising in the applications of imaging steep subsalt reflectors and overhanging salt flanks.
In this section an example will be taken to show the good performance of the method in imaging salt flanks.

Figure 8(a) shows the velocity model in which the salt bulk with overhanging flanks is surrounded by the background $c(z)$ media. The location number of the single source is 130 and the receivers are distributed from No. 30 through 426 in Figure 8(a). The surface records we used for imaging are shown in Figure 8(b). These records are generated by the FD method. Figure 8(c) shows the image obtained from these surface records using the new propagator. The boundaries of the salt dome including the whole left flank are clearly imaged. Even part of the right flank is also shown on the image. Without turning waves, the upper part of the overhanging left flank is hard to be imaged. This is an image using only one shot. The direct waves contaminate the image considerably. Other artifacts may be produced by the edge effects. With more data from more shots, we expect to get much more clear image of the salt flanks.

Conclusions

It is shown that the proposed one-way wave propagation method by combining two orthogonally propagated one-way waves with wavefront reconstruction can overcome the angle limitation of regular one-way methods. The proposed method has accurate super-wide angle (greater than 90º) propagation and can model turning waves. Numerical examples demonstrated that the new propagator can be used in imaging steep subsalt reflectors and overhanging salt flanks.

Acknowledgements

We thank Dr. Xiao-Bi Xie for the program help and many discussions. The supports from the WTOPI (Wavelet Transform On Propagation and Imaging for seismic exploration) Research Consortium and the DOE/BES Project at University of California, Santa Cruz are acknowledged.

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One way boundary element modeling and salt multiples

Yaofeng He and Ru-shan Wu

Abstract

The one-way elastodynamic boundary element modeling method (BEM) is developed to calculate the primary transmission/reflection arrivals in 2D P-SV case for inclusion model with strong velocity contrast. The primaries are then subtracted from the whole wave field to obtain the internal multiples. The primary reflection waves from top and bottom interfaces of a lens model and the multiples between these two interfaces are studied. Numerical examples demonstrate that it is efficient to obtain the primary transmission/reflection waves generated by internal interfaces with strong velocity contrast by this technique.

Introduction

One way and one-return (multiple-forward-single-backscattering approximation) propagator has shown its high efficiency in modeling the primary transmission/reflection signals and has been widely used in seismic migration and imaging (e.g. Wu, 1994; Xie and Wu, 2001; Wu and Wu, 2006). However, one return modeling fails when the velocity perturbation of the heterogeneity increases up to more than 40% of the background. For the SEG-EAGE model, the velocity perturbation may be more than 200%. The high velocity contrast lead to strong internal multiples, which can significantly contaminate the reflection signals from structures beneath the inclusion. How to efficiently eliminate these multiples is still a challenging problem. The boundary element method is widely implemented in full seismic wave field modeling. It is efficient in solving the forward modeling problems especially for the rough topographical scattering and for the inclusion model with strong velocity contrast. Sánchez-Sesma and Campillo (1991) applied the indirect boundary element method to P-SV wave scattering problem. Fu and Wu (2001) developed a SH wave field connection technique to model Lg wave in long regional waveguides. This technique leads to significant computational savings in time and memory compared with the whole wave guide boundary element method. Ge et al. (2005) extended this connection technique to apply to P-SV case for long range regional wave field simulation.

Our goal is to develop a technique of modeling primary reflections in heterogeneous media with strong contrast inclusions using boundary element method. For stratified media, we can partially decouple the interactions between some elements in the full boundary element method and hence obtain the primary transmission/reflection waves. As for the salt dome model, we can separate the dome into top and bottom parts by adding two artificial interfaces. We can therefore decouple the interaction between the upper interface and the bottom interface to get the primary transmission/reflection waves. By subtracting these transmission/reflection waves from the whole wave field, we can obtain the multiples purely generated by interaction of the upper and bottom interfaces of the salt dome. These synthetic multiples can be used to predict the multiples in real data. By combining the advantage of BEM and one-way/one-return propagator together,
the one-way BEM is expected to efficiently handle the internal interface scattering problem due to strong velocity contrast inclusion.

**One-Way Boundary Element Method**

With the aid of free space Green’s function, the partial differential equation for elastic wave can be transformed into a boundary integral equation for homogeneous media or a volume integral equation for inhomogeneous media. Based on the representation theorem (Aki and Richards, 1980), the boundary integral equation for the displacement in each domain satisfies:

\[
c(r)u(r) + \int_\Gamma [u(r')\Sigma(r, r') - T(r')G(r, r')]d\Gamma(r') = \int_\Omega f(r', \omega)G(r, r')d\Omega(r')
\]

(1)

where \( u(r) \) is the displacement vector, \( T(r) \) is the traction vector. The coefficient \( c(r) \) generally depends on the local geometry of the boundary. \( G(r, r') \) and \( \Sigma(r, r') \) are the fundamental solutions (Green tensors) for displacement and traction, respectively. Usually the right side of the equation can be viewed as the incident wave. By discretizing the boundaries into linear elements, we can express the boundary integral equation in matrix form:

\[
HU + GT = U_0
\]

(2)

where \( H \) and \( G \) are the matrix made up of the integrals over Green’s functions on each element. Once the boundary values are obtained by solving this linear matrix equation, the displacement of any point inside the domain can be represented by the boundary integral via equation (1). The most time-consuming part of the full wave boundary element modeling is to solve a 4N*4N full rank matrix for 2D P-SV case, where N is the number of all the nodes.

To interpret the idea of one-way boundary element method, consider 2D steady-state elastic wave propagation in a simplified stratified model depicted in figure 1. Here we aim to develop an interface-by-interface approach to obtain the primary transmitted wave through \( \Gamma \) in domain \( \Omega \). First, we calculate the transmitted wave through \( \Gamma_1 \) by applying the full-wave BEM to domain \( \Omega_1 \) and domain \( \Omega_2 \) with interface \( \Gamma_1 \), ignoring the effect of interface \( \Gamma_2 \). The diffraction from truncated edges can be suppressed using infinite boundary elements. The second step is to propagate the output wave field to the interface \( \Gamma_2 \) as the incident wave. Finally we obtain the transmitted wave through interface \( \Gamma_2 \) in domain \( \Omega_3 \) by solving the two-domain boundary value problem involving domain \( \Omega_2 \) and domain \( \Omega_3 \) with interface \( \Gamma_2 \). Each time we only need to solve a much smaller matrix associated with the current interface rather than a much larger full rank matrix in full-wave BEM. This leads to much savings in computational time and memory. However, this technique decouples the wave field interaction between interface \( \Gamma_1 \) and \( \Gamma_2 \), thus eliminates the multiples between \( \Gamma_1 \) and \( \Gamma_2 \), and can only obtain the primary transmitted waves. Therefore we call this technique ‘one-way’ boundary element modeling. For the model with inclusion such as the SEG-EAGE salt model, we can add two artificial interfaces to separate the inclusion into two adjacent domains and can implement the one-way boundary element method to calculate the primary transmitted wave field under the salt dome. The primary reflection wave field on the surface \( \Gamma_1 \) can also be calculated by propagating the backward scattering wave field upward to the surface. This one-way boundary element method is expected to efficiently handle the scattering problem due to the internal interface with strong velocity contrast.
Numerical Examples

In this section, numerical examples are presented to test the feasibility of the one-way boundary element method. We use a model with a Gaussian-shaped lens embedded in an otherwise homogeneous media. The parameters for the lens model are \( V_p = 2.5 \text{ km/s} \), \( V_s = 1.5 \text{ km/s} \), \( \rho = 2.0 \text{ g/cm}^3 \) for the outer homogeneous domain, \( V_p = 5.0 \text{ km/s} \), \( V_s = 3.0 \text{ km/s} \), \( \rho = 2.2 \text{ g/cm}^3 \) for the lens inclusion. The top and bottom interfaces of the lens are Gaussian type
\[
h = \pm h_0 e^{-(x-x_0)^2/\lambda^2}
\]
with \( h_0 = 3.0 \text{ km} \), \( \lambda = 3.0 \text{ km} \). Artificial interfaces are added at the edge to separate the domains of interaction. The sketch of the model and geometry of the source and receivers are shown in figure 2. Figure 3 shows the synthetic seismograms calculated by full-wave BEM and finite difference (FD) method. The good agreement of the results suggests that the artificial zero-distance interface has little effect on the wave field. The seismograms calculated by one-way boundary element method are shown in figure 4. The primary reflections from the upper and bottom interfaces of the lens are shown in figure 5 and figure 6, respectively. Finally, we can subtract the primary reflections due to the interfaces from the full wave field to obtain the multiples generated between the top and bottom interfaces of the lens, which is shown in figure 7. The multiples are quite complicated because of the lens’ focusing effect on seismic waves.

Acknowledgement

We thank Xiao-Bi Xie, Jun Cao and Yingcai Zheng for their valuable discussions and suggestions.

References


Figure 1: The layered model used to interpret the idea of one-way boundary method for seismic wave forward modeling. Triangles indicate the receivers.

Figure 2: The lens model and the geometry of the source and receivers. The source is located right above the lens indicated by the star, the receivers by triangles.
Figure 3: The wave field calculated by the full-wave BEM (dash) and FD method (solid).
Figure 4: Synthetic seismograms by one-way BEM (dash) and full-wave BEM (solid) for the model in figure 2.
Figure 5: Primary reflections from the upper interface of the lens.
Figure 6: Reflection from the bottom interface of the lens.
Figure 7: The multiples between the top and bottom interfaces of the lens.
PART IV

SOME PUBLISHED AND ACCEPTED PAPERS
Target-oriented beamlet migration based on Gabor-Daubechies frame decomposition

Ling Chen, Ru-Shan Wu, and Yong Chen

ABSTRACT

We develop beamlet propagation and imaging using Gabor-Daubechies (G-D) frame decomposition based on local perturbation theory and apply it to target-oriented prestack depth migration. The method is formulated with local background velocities and local perturbations in wavefield extrapolation. The localized propagators and phase-correction operators are obtained analytically or semianalytically by one-way operator decomposition and screen approximation in the beamlet and space-beamlet mixed domain. Beamlet wavefields have superior localization properties in both local space and direction (wavenumber) over Gaussian beams in the sense that localizations are not limited within short propagation distances in either homogeneous or heterogeneous media. Comparisons of the prestack depth-migrated images for the 2D SEG-EAGE salt model and the Marmousi model indicate that, for seismic-wave propagation and imaging in complex structural environments, the G-D beamlet propagator has higher accuracy and better wide-angle properties than do global propagators. Target-oriented prestack G-D beamlet migration is performed by means of local-angle-domain imaging and controlled superposition of common-angle images based on the local directivity features of the target structures. This considers the spatial and direction localizations of beamlets. As a numerical example, we process the 2D SEG-EAGE salt-model prestack data. The results show that the proposed migration method has considerable advantages in suppressing noise and enhancing structural features. Image quality for subsalt structures, especially for steep faults, is improved through structure-based superposition of common-angle images. This demonstrates the potential and capability of beamlet migration in target-oriented seismic imaging.

INTRODUCTION

Various techniques and methods for wavefield decomposition, propagation, and migration imaging have been developed and applied to seismic exploration, including the Kirchhoff high-frequency asymptotic method (Schneider, 1978; Gray and May, 1994; Audebert et al., 1997), the wave-equation phase-shift method (Giraud, 1978; Stolt, 1978), the phase-screen method (Stoffa et al., 1990; Wu and Huang, 1992), and the generalized screen propagator (GSP) method (de Hoop et al., 2000; Xie et al., 2000; Jin et al., 2002; Wu, 2003). In these methods, the wavefield is expanded by sets of basic functions such as spatial Fourier harmonics, modes, and spatial Green’s functions. The common factor in these basic functions is their global nature in either the space or the wavenumber domain. Basic functions in the wavenumber domain are plane waves, which have the best localization in direction but no spatial localization. Basic functions in the space domain are point sources (delta functions), which have the best localization in space but have no direction localization. In highly heterogeneous media, the global nature of these two kinds of basic functions creates some difficulties in correcting for local heterogeneities, which is necessary to improve image quality and to extract information in the target area.

To overcome this fundamental limitation caused by the global nature of these propagators, efforts have been made to investigate and develop wavefield decomposition and extrapolation methods with localization in both space and direction. High-frequency, ray-based beam-summation methods, such as the complex source-generated beam and the Gaussian
beam methods (Deschamps, 1971; Červený et al., 1982; Felsen et al., 1991; Hill, 1990, 2001), have been developed to solve the caustic problem encountered in the traditional geometric ray method and, more importantly, to achieve wavefield extrapolation in a localized way. High-frequency asymptotics and paraxial approximation, however, inevitably limit the accuracy of these methods and their applicability to complex media. Steinberg (1993) and Steinberg and Birman (1995) derive a wave-equation-based localized phase-space propagator using windowed Fourier transforms (WFTs) and a perturbation approach. The localized propagator depends mainly on the local heterogeneity of the medium and is much easier to construct with high accuracy than are global propagators. The perfect WFT decomposition and reconstruction, however, is formally expensive, so the method is difficult to adopt for practical use. In addition, it is formulated with a global perturbation to the partial differential wave equation, which, for strong lateral variations, may result in large perturbations and might produce significant errors in wavefield propagation. In window screen-propagator methods (Wu and Jin, 1997; Jin and Wu, 1999), local background velocities and local perturbations are introduced through WFT. Although this provides a possible way to improve the accuracy of wavefield extrapolation, the broad overlapping windows in WFT and the empirical interpolations employed limit the utility of this method.

Advanced mathematical techniques, especially the newly developed wavelet transform and the more general frame theory (Daubechies, 1990, 1992; Mallat, 1998), provide a solid foundation for the development of localized propagators. Wu et al. (2000) and Wu and Chen (2001) propose a wave-equation-based wavefield extrapolation formulation based on local perturbation theory. They derive analytically the localized propagators by using the Gabor-Daubechies (G-D) frame (Appendix A) to decompose and propagate the wavefield in the beamlet domain. The basic functions (vectors) of the G-D frame decomposition are Gaussian windowed (Gaussian-windowed) harmonic wavefields. Applying G-D frame decomposition to the wavefield in a local homogeneous region is like decomposing the wavefield into many small Gaussian beams (beamlets) at different window locations and with different beam directions. The G-D frame decomposition of wavefields is redundant, as measured quantitatively by the redundancy ratio $R$ (see Appendix A) — a condition for stable reconstruction and hence somewhat more expensive than orthogonal decomposition schemes. The spatially and directionally confined frame vectors, however, result in many desirable features of the G-D beamlet wavefield extrapolation method. With local background velocities and local perturbations adopted in each extrapolation step, the G-D beamlet propagator propagates the beamlets step by step in depth, accounting for both individual beamlet propagation and cross-coupling between beamlets through local heterogeneities during propagation.

The G-D beamlet propagator is more flexible and has better wide-angle properties than do traditional global propagators, as manifested by numeric results from forward modeling and poststack migration (Wu and Chen, 2001, 2002a). Moreover, the localization properties in both space and direction (wavenumber) of G-D beamlets can be used to extract local and directional information of the wavefields and reflections during the migration procedure. This is particularly desirable for target-oriented imaging and various studies on fine-scale structures and angle-domain analyses.

Here we give a detailed description of the G-D beamlet wavefield extrapolation and migration method and apply it to target-oriented imaging. For clarity, we list in Table 1 the major notations. The feasibility of using a G-D beamlet propagator for prestack depth migration is tested through comparative studies of the images produced by beamlet propagators versus global propagators for both the SEG-EAGE salt model 2D prestack data set (Aminzadeh et al., 1994; Aminzadeh et al., 1995; SEG/EAGE 3-D Modeling Committee, 1994) and the Marmousi model data set (Versteeg and Grau, 1991). As a numeric example of target-oriented imaging, we construct local-angle image matrices and the corresponding common-angle-snap image (CDAI) gathers for the 2D SEG-EAGE salt model. We perform structure-based partial summation of the CDAI gathers to improve the image quality of the targets.

### GABOR-DAUBECHIES BEAMLET WAVEFIELD EXTRAPOLATION

For demonstration purposes, we limit our derivation to the 2D $(x, z)$ case. Generalization to the 3D case is straightforward but involves more technical considerations for practical implementation. In the frequency-space domain, the scalar wave equation can be written as

$$\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial x^2} + \frac{\omega^2}{v^2(x, z)} u(x, z, \omega) = 0,$$

where $\omega$ denotes the circular frequency; $u(x, z)$ is the space-domain velocity field, and $u(x, z, \omega)$ represents the pressure field in the frequency-space domain.

With the G-D frame vectors, the wavefield at depth $z$ can be decomposed into beamlets (frame vectors) with windows along the $x$-axis (see Appendix A):

$$u(x, z, \omega) = \sum_m \sum_n \left[ \hat{u}(n, \omega) g_{mn}(x) \right] g_{mn}(x),$$

where $u_{mn}(z, \omega)$ is the beamlet decomposition coefficient. The corresponding frame vector $g_{mn}(x)$,

$$g_{mn}(x) = g(x - n\Delta_x) e^{im\Delta_z},$$

is a Gaussian-windowed harmonic function with space locus $\Delta_x = n\Delta_x$ and wavenumber locus of $\Delta_{\omega} = m\Delta_{\omega}$. Here, $\Delta_x$ and $\Delta_{\omega}$ stand for the space and wavenumber sampling intervals of the frame vectors, respectively. $g_{mn}$ is the dual frame vector of $g_{mn}$, constructed in a same way as $g_{mn}$ but with a dual-window function $g(\tilde{x})$ instead of the Gaussian window $g(x)$ in equation 3. Dual frame vectors are required for redundant G-D frame decomposition and reconstruction. Detailed descriptions of the G-D frame parameters, selection of the
dual-window function, and calculation of the decomposition coefficients can be found in Appendix A.

Letting \( G^{u*}(x, z + \Delta z, \omega) \) represent the Green’s function (space-domain beamlet propagator) describing the wavefield distribution at depth \( z + \Delta z \) generated by the beamlet \( g_{\text{ref}}(x) \) at depth \( z \) in an arbitrary medium, we can reconstruct the wavefield at \( z + \Delta z \) by superimposing contributions from all of the beamlets:

\[
u(x, z + \Delta z, \omega) = \sum_{m} \sum_{n} u_{mn}(x, \omega) G^{u*}(x, z, \Delta z, \omega).
\]

(4)

We can further decompose \( G^{u*} \) at \( z + \Delta z \) into beamlets to get the wavefield of the Green’s function in the pure beamlet domain:

\[
G^{u*}(x, z, \Delta z, \omega) = \sum_{j} \sum_{i} (G^{u*}(x, \xi_{j}), g_{j}(x)) g_{j}(x),
\]

(5)

where \( P^{u*}_{ij}(z, \Delta z, \omega) \) stands for the Green’s function (beamlet propagator) in the beamlet domain. Substituting equation 5

Table 1. Notations used in G-D frame representation and derivation for G-D beamlet wavefield extrapolation and migration.

<table>
<thead>
<tr>
<th>Notations</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>G-D frame</td>
<td>Space sampling interval of frame vectors</td>
</tr>
<tr>
<td>( \Delta_x )</td>
<td>Wavenumber sampling interval of frame vectors</td>
</tr>
<tr>
<td>( g(x), \mathcal{F}(\xi) )</td>
<td>Gaussian window function and its Fourier transform</td>
</tr>
<tr>
<td>( \mathcal{F}(\xi) )</td>
<td>Dual-window function and its Fourier transform</td>
</tr>
<tr>
<td>( \xi_m )</td>
<td>Space locus ((= m \Delta_x)) of the nth window</td>
</tr>
<tr>
<td>( g_{\text{ref}}(x) )</td>
<td>G-D frame vector with space locus of ( x_0 ) and wavenumber locus of ( \xi_m )</td>
</tr>
<tr>
<td>( g_{\text{ref}}(x) )</td>
<td>Dual frame vector of ( g_{\text{ref}}(x) )</td>
</tr>
<tr>
<td>( N_m )</td>
<td>Number of windows in decomposition</td>
</tr>
<tr>
<td>( N_x )</td>
<td>Number of local wavenumbers in each window</td>
</tr>
<tr>
<td>( R )</td>
<td>Redundancy ratio</td>
</tr>
</tbody>
</table>

G-D beamlet extrapolation

- \( u(x, z, \omega) \): Wavefield at depth \( z \)
- \( u_{mn}(x, \omega) \): G-D frame decomposition coefficients of \( u(x, z, \omega) \)
- \( G^{u*}(x, z, \Delta z, \omega) \): Space-domain Green's function describing the beamlet \( g_{\text{ref}}(x) \) evolving from depth \( z \) to depth \( z + \Delta z \)
- \( \nu(x, z) \): Velocity field in the space domain
- \( k(x, z) \): Actual wavenumber
- \( \tilde{\nu}(x, \xi) \): Local reference velocity in the \( m \)th window
- \( k_{\xi}(\xi_m, z) \): Local background wavenumber with velocity \( \tilde{\nu}(\xi_m, z) \) and transverse wavenumber \( \xi \)
- \( \Delta \) | Square-root operator |
- \( \mathcal{I}(\xi, \omega) \): Wavenumber-domain free thin-slab propagator |
- \( \mathcal{I}(\Delta \xi, \omega) \): Space-domain perturbation operator |
- \( P^{u*}_{m}(z, \Delta z, \omega) \): Beamlet-domain propagator |
- \( P^{u*}_{ij}(z, \Delta z, \omega) \): Beamlet-domain free propagator |
- \( u_{mn}(x, \omega) \): Phase-corrected G-D frame decomposition coefficients of \( u(x, z, \omega) \)

Prestack G-D beamlet migration

- \( u^0(x, z, \omega) \): Incident field from source
- \( u_{mn}^{\text{D}}(x, \omega) \): G-D frame decomposition coefficients of \( u^0(x, z, \omega) \)
- \( u^0(x, z, \omega) \): Scattered field from receivers
- \( u_{mn}^{\text{D}}(x, \omega) \): G-D frame decomposition coefficients of \( u^0(x, z, \omega) \)
- \( \theta_{\xi} \): Incident angle corresponding to the local wavenumber \( \xi_m \)
- \( \theta_j \): Scattering angle corresponding to the local wavenumber \( \xi_j \)
- \( \theta_a \): Normal (dip) of the local reflector
- \( \theta_r \): Reflection angle of the local reflector
- \( u^a(x, z, \xi_m, \omega) \): Directional incident wavefield at points \((x, z)\) in terms of local wavenumber \( \xi_m \) and incident angle \( \theta_a \)
- \( u^{\text{D}}(x, z, \xi_j, \omega) \): Directional scattered wavefield at points \((x, z)\) in terms of local wavenumber \( \xi_j \) and scattering angle \( \theta_r \)
- \( I(\theta_a, \theta_j, x, z) \): Local-angle image matrix at point \((x, z)\) in terms of incident-scattering angle pair \((\theta_a, \theta_r)\) and normal-reflection angle pair \((\theta_a, \theta_r)\)
- \( I(\theta_a, x, z) \): Common reflection-angle image
- \( I(\theta_a, x, z) \): Common dip-angle image
- \( I(x, z) \): Image of total strength from prestack depth migration
into the right-hand side of equation 4 yields

\[ u(x, z + \Delta z, \omega) = \sum_{m} \sum_{n} u_{mn}(z, \omega) \sum_{j} p_{ij}^{\text{mn}}(z, \Delta z, \omega) g_{ij}(x) \]

\[ = \sum_{j} \left[ \sum_{m} \sum_{n} p_{ij}^{\text{mn}}(z, \Delta z, \omega) u_{mn}(z, \omega) \right] g_{ij}(x). \]  

(6)

Since the G-D frame representation of the wavefield \( u(x, z + \Delta z, \omega) \) can be expressed as

\[ u(x, z + \Delta z, \omega) = \sum_{j} \sum_{i} u_{ij}(z + \Delta z, \omega) g_{ij}(x), \]  

(7)

we have

\[ u_{ij}(z + \Delta z, \omega) = \sum_{m} \sum_{n} p_{ij}^{\text{mn}}(z, \Delta z, \omega) u_{mn}(z, \omega). \]  

(8)

Therefore, \( p_{ij}^{\text{mn}}(z, \Delta z, \omega) \) will serve as the wavefield propagator in the beamlet domain, which enables calculation of both individual beamlet propagation and the crosscoupling in propagation between different beamlets.

From equations 6 and 8, we see that the accuracy of the reconstructed wavefield depends on the accuracy of the beamlet propagator \( p_{ij}^{\text{mn}}(z, \Delta z, \omega) \). For large lateral velocity contrasts, the localized beamlet decomposition of wavefields allows us to deal with the lateral velocity variations locally when constructing the beamlet propagator. First, we introduce a background velocity for each beamlet window. For example, in the \( n \)th window the background velocity \( v(x_n, z) \) is selected usually as the average or minimum velocity within the window. In this way, local-perturbations can be calculated from the following relation:

\[ k^2(x, z) = k_0^2(x_n, z) + [k^2(x, z) - k_0^2(x_n, z)], \]  

(9)

where \( k(x, z) = \omega/\sqrt{v(x, z)} \) is the actual wavenumber and \( k_0(x_n, z) = \omega/\sqrt{v(x_n, z)} \) is the local background (reference) wavenumber. The second term on the right-hand side of equation 9 gives the corresponding local perturbations. Substituting field-decomposition equation 2 and local-perturbation equation 9 into wave-equation 1 yields the beamlet-domain wave equation,

\[ \sum_{n} \left( \partial^2_{x} + A_n \right) \sum_{m} u_{mn}(z, \omega) g_{mn}(x) = 0, \]  

(10)

where \( A_n \) is the square root operator:

\[ A_n = \sqrt{\partial^2_{x} + k_0^2(x_n, z) + [k^2(x, z) - k_0^2(x_n, z)]}. \]  

(11)

Then we invoke the one-way wave approximation, which neglects interactions between the forward-scattered and backscattered waves, and derive a formal solution for the evolution of beamlets (windowed wavefields):

\[ G^{\text{mn}}(x, z, \Delta z, \omega) = e^{i \xi(x_n, z)} g_{mn}(x). \]  

(12)

Here, \( e^{i \xi(x_n, z)} \) represents a thin-slab propagator (Wu and de Hoop, 1996; de Hoop et al., 2000) for the beamlets under an assumption of vertical homogeneity within the thin slab of \( \Delta z \) in thickness. Such a vertical-homogeneity assumption is commonly adopted in various recursive depth-wavefield extrapolation algorithms, such as those primarily implemented in the frequency-space domain (Holberg, 1988; Blacquière et al., 1989). The sign of the superscript corresponds to upward (−) or downward (+) propagation. For simplicity, we drop the superscript hereafter. The square root operator \( A_n \), which is actually a pseudodifferential operator, can be approximated in the same manner as in the expansion of GSPS (Xie and Wu, 1998; de Hoop et al., 2000; Jin et al., 2002; Wu, 2003).

We adopt a local perturbation scheme in which the reference velocity varies with the window, instead of using a global background velocity as in GSPS methods. Since beamlets have significant values only within neighboring windows, and their spatial spreads are small for short propagation distances, the term \( k^2(x, z) - k_0^2(x_n, z) \) in equations 9 and 11 can be treated as a small perturbation. Then, the first-order expansion with a small-angle approximation is sufficient to ensure high accuracy and can yield an approximate expression for \( A_n \). As a result, \( A_n \) can be written as a sum of background and perturbation parts (see Appendix B)

\[ A_n(x, z) \approx \sqrt{\partial^2_{x} + k_0^2(x_n, z) + [k(x, z) - k_0(x_n, z)]} \]

\[ = \sqrt{\partial^2_{x} + k_0^2(x_n, z) + \Delta k_n(x, z)}. \]  

(13)

Accordingly, the thin-slab propagator \( e^{i \xi(x_n, z)} \) is separated into two parts: a wavenumber-domain-free propagator \( e^{i \xi(x_n, z)} \Delta k_n(x, z) \), accounting for one-way wave propagation through the local homogeneous medium, and a space-domain perturbation operator \( e^{i \xi(x_n, z)} \Delta k_n(x, z) \), accounting for the phase correction for local velocity perturbations (Appendix B). For the free propagator \( e^{i \xi(x_n, z)} \Delta k_n(x, z) \), \( \xi(x_n, z) = \sqrt{k_0^2(x_n, z) - \xi^2} \) is the local vertical wavenumber corresponding to a transverse wavenumber \( \xi \) and the local background wavenumber \( k_0(x_n, z) \).

Following the decomposition of the thin-slab propagator, the Green’s function \( G^{\text{mn}} \) in equation 12 can be constructed as

\[ G^{\text{mn}}(x, z, \Delta z, \omega) = G_0^{\text{mn}}(x, z, \Delta z, \omega) e^{i \xi(x_n, z) \Delta z}. \]  

(14)

Applying spatial Fourier transforms to \( G_0^{\text{mn}}(x, z, \Delta z, \omega) \) results in

\[ \hat{G}_0^{\text{mn}}(\xi, z, \Delta z, \omega) = e^{i \xi(x_n, z) \Delta z} \hat{g}_{mn}(\xi), \]  

(15)

where \( \hat{G}_0^{\text{mn}}(\xi, z, \Delta z, \omega) \) and \( \hat{g}_{mn}(\xi) \) are the spatial Fourier transforms of \( G_0^{\text{mn}}(x, z, \Delta z, \omega) \) and \( g_{mn}(x) \), respectively. The corresponding beamlet propagator defined in equation 5 is then decomposed into a free propagator and a perturbation operator. The G-D beamlet-domain free propagator can be expressed analytically (for details, see Appendix C) as

\[ P_0^{\text{mn}}(z, \Delta z, \omega) = \langle \hat{G}_0^{\text{mn}}(\xi, \Delta z, \omega) \rangle \]

\[ = \frac{1}{2\pi} e^{-i \xi(\xi x_n - z s_0)} \int \frac{d\xi \hat{g}(\xi - \xi_0)}{\xi s(x_n - \xi_0)} e^{i \xi(x_n, z) \Delta z}, \]  

(16)

where \( \hat{g}(\xi) \) and \( \hat{g}(\xi) \) are the Fourier transforms of the Gaussian window \( g(x) \), and the dual-window \( \hat{g}(\xi) ; x_n, \xi_0, s_0 \) are the \( nth \) and \( nth \) window locations (centers), and \( \xi_0, \xi_n \) are the \( mth \) and
nth local wavenumbers of the beamlets. The asterisk (\(^\ast\)) stands for complex conjugate.

For the perturbation operator, we adopt the first-order approximation (phase-screen correction) in the local space-beamlet mixed domain and derive the phase-corrected beamlet coefficients (Appendix D):

\[ u_{mn}^p(x, z, \omega) = \int dx u(x, z, \omega) e^{i\Delta k(x, z, \omega)z} \times \tilde{g}(x - \tilde{x}_n) e^{-i\tilde{\omega}x}, \]  

where the superscript \( p \) denotes phase correction. The wavefield in the space domain is then constructed as

\[ u(x, z + \Delta z, \omega) \approx \sum_j \sum_l \left[ \sum_n \sum_m P_{jm}^{mn}(z, \Delta z, \omega) \right. \times \left. u_{mn}^p(z, \omega) \right] g_{jl}(x). \]

With the propagator decomposition mentioned above, one can achieve wavefield extrapolation by first applying the perturbation operation in the mixed domain and then performing the wavefield propagation in the beamlet domain with the local background medium. For local perturbations, the perturbation correction is relatively small and the free propagation plays a dominant role in the procedure.

In addition to frequency, the redundancy ratio, which quantitatively measures the redundancy of the G-D frame (see Appendix A), is an important factor influencing the properties of the beamlet-domain free propagator. Figures 1a–d show the G-D beamlet free-propagator matrices in a homogeneous medium for different frequencies (\( f = 5, 25 \text{ Hz} \)) and different redundancy ratios (\( R = 2, 4 \)). In the presentation, the elements in a 4D propagator matrix \( \{P_{jm}^{mn}; 0 \leq j < N_f, 0 \leq l \leq N_c, 0 \leq m \leq N_b, 0 \leq n \leq N_b\} \) are sorted into a 2D matrix \( P_{jl}(r, c) \), of which the row index \( r = (l - 1) \times N_c + j \) and the column index \( c = (n - 1) \times N_b + m \) represent the beamlet indices for input and output beamlets of the one-step propagation, respectively. Only those elements that are larger than the specific threshold (0.1% of the largest element) are shown and considered as effective elements in the implementation of beamlet wavefield extrapolation. The ratios of effective element number to total element number of the propagator matrices are about 7%, 10%, 5%, and 6% for (a), (b), (c), and (d), respectively.

We see that the free propagators are all highly sparse matrices, the sparsity becoming higher as the redundancy ratio increases but decreasing with frequency. Although higher redundancy ratios result in sparser propagator matrices and lead to better localizations of the wavefield (Appendix A), the rapidly increased element number of the propagator matrix, which is proportional to the square of the beamlet coefficient number and thus to the square of the redundancy ratio — and the associated computation costs — prohibit efficient

![Figure 1](https://via.placeholder.com/150)

Figure 1. (a)–(d) Matrix representations of G-D beamlet free propagators in a homogeneous medium with different frequencies and different redundancy ratios. (a) \( f = 5 \text{ Hz}, R = 2 \); (b) \( f = 25 \text{ Hz}, R = 2 \); (c) \( f = 5 \text{ Hz}, R = 4 \); (d) \( f = 25 \text{ Hz}, R = 4 \). For all cases, 32 windows \( (N_b = 16) \) with a length of 64 samples of wavefields in the x-direction are used; within each window, the numbers of local wavenumbers are set as \( N_f = 16 \) and \( N_b = 32 \) for \( R = 2 \) and 4, respectively. In each panel, only the window indexes \((l, n) \) are given for both the row and column coordinates; there are \( N_b \) elements of different local wavenumbers in each window. The scale represents the amplitude of the elements. (e)–(h) Element distributions within two window blocks marked in the propagator matrices in (a) and (b). (e), (f) For the diagonal block, presenting self- and cross-coupling of beamlets within one window. (g), (h) for the off-diagonal block, showing cross-coupling between beamlets at two windows of different locations.
Target Oriented Beamlet Migration

Figure 2: Two inhomogeneous models. (a) A three-layered model whose middle layer is heterogeneous with laterally varying velocities. (b) The 2D SEG-EAGE salt model (SEG/EAGE 3D Modeling Committee, 1994; Aminzadeh et al., 1994, 1995).

implementation of beamlet propagation. From our numeric experiments, a redundancy ratio of two has moderate computation expense yet maintains good localization of the wavefield. Therefore, \( K \) is always set to a value of two in the following numerical tests.

The high concentration of the elements within the diagonal band of the free beamlet propagators in Figures 1a-d indicates the propagation coupling of beamlets is significant only between neighboring windows. To investigate in detail the beamlet coupling between different wavenumber, we pick two window blocks: one on the diagonal and the other off diagonal from the propagator matrices of \( K = 2 \) (marked as rectangles in Figures 1a, b). The element distributions of the two blocks are shown in Figures 1e-h. Figures 1e and 1f correspond to the self- and cross-coupling of beamlets within one window (diagonal block), and Figures 1g and 1h illustrate the cross-coupling of beamlets from different windows (off-diagonal block). For all cases, coupling occurs only between beamlets of adjacent wavenumbers as manifested by the concentration of elements along the diagonal or at the corner of the minor diagonal.

We summarize our G-D beamlet wavefield extrapolation scheme as follows. At the beginning of the procedure, the parameters of the G-D frame — such as the spacing intervals \( \Delta_x \) and \( \Delta_z \), the numbers of windows and local wavenumbers in each window \( N_x \) and \( N_z \), etc. — and the length and shape of the Gaussian window \( g(x) \) and its dual-window \( g(z) \) — are determined based on the frame construction principle and the procedure described in Appendix A as well as according to the structural properties of the model considered. At each depth, a local phase-screen correction is applied to the space-domain wavefield \( u(x, z, \omega) \), and the corrected wavefield is decomposed into beamlets based on equation 17 to produce the modified beamlet coefficients \( \hat{a}^{\omega}_{m} \). Then the beamlet free propagator \( F_{n}^{\omega} \) is calculated from equation 16. Finally, the wavefield at the next depth is obtained by substituting \( \hat{a}^{\omega}_{m} \) and \( F_{n}^{\omega} \) into equation 18. Such a three-step procedure is carried out iteratively to construct the wavefield for the whole space.

PROPAGATION OF SINGLE G-D BEAMLLET SOURCES AT THE SURFACE

Using the beamlet-domain wavefield extrapolation algorithm, through forward modeling we investigate the propagation features of individual G-D beamlet sources excited on the surface. Three velocity models are considered here: a homogeneous model, a layered model with one layer of laterally varying velocities sandwiched by two homogeneous layers (Figure 2a), and the complex SEG/EAGE 2D salt model containing a high-velocity salt body with complicated top surface (Figure 2b). The wavefields of single beamlet sources with different frequencies are shown in Figures 3-5 for the three models. With the well-constructed beamlet propagator, the wavefields are propagated accurately, even for
the highly heterogeneous case (Figure 5). The beamlet-source fields clearly show space expansion along the propagating directions and exhibit severe wavefront distortion in accordance with the velocity jump at sharp boundaries. The higher the frequency, the narrower the spread of the wavefield. The wavefront distortion, in contrast, appears to be of comparable degree at different frequencies (Figures 4 and 5).

In some beam-based wavefield extrapolation techniques, such as the Gaussian beam method (Cerveny et al., 1982; Nowack and Aki, 1984; Hill, 1990, 2001), the wavefield is decomposed into beams only at the initial depth (the surface), and the beams are extrapolated directly to the model space without iterative wavefield decomposition and reconstruction. For a large extrapolation distance, the initially localized beams gradually lose their localizations. For the Gaussian beam method, which uses the parabolic approximation and the ray approximation in wavefield extrapolation, the beam distortion and abnormal expansion would become more severe along propagating paths. In the beamlet method, the space-domain wavefield is decomposed into beamlets, propagated in the beamlet domain (plus a phase-correction in the local space-beamlet mixed domain), and reconstructed from beamlets at every depth. In this respect, the beamlet method should be more accurate for wavefield propagation. More importantly, the method allows a better means of extracting local directional information of the wavefield at each extrapolation step. This feature is particularly advantageous for directivity-involved analysis and angle-domain local structural imaging.

**PRESTACK G-D BEAMLET MIGRATION FOR THE 2D SEG-EAGE SALT MODEL AND THE MARMOUSI MODEL**

Poststack G-D beamlet depth migration has been conducted by Wu and Chen (2001, 2002a) to show the validity and feasibility of G-D beamlet propagators applied to seismic imaging. In this paper, we perform common-shot prestack beamlet migration for the 2D SEG-EAGE salt model and the Marmousi model. While the former model is characterized by a strong velocity contrast and large dips of subsalt faults, the latter model is structurally complex, with many thin layers broken by several major faults and an unconformity surface.

Considering the high velocity contrast between the salt body and the surroundings, we choose narrower windows in the beamlet decomposition of wavefields for the SEG-EAGE salt model than those for the Marmousi model to reduce lateral perturbations in each window. The parameters of the G-D beamlets used as well as the information for both of the models and the prestack data sets are listed in Table 2. In common-shot migration, the source field \( u^s(x, z = 0, \omega) \) for each shot is forward-propagated, and the corresponding recorded data field \( u^r(x, z = 0, \omega) \) is back-propagated to the image space in accordance with equation 18. Here, \( S \) is the shot index and \( R \) stands for the receiver array for shot \( S \). At each depth, the traditional space-domain image condition (Claerbout, 1971)

![Figure 4. Same as Figure 3 except for a three-layered model (Figure 2a).](image)

![Figure 5. Same as Figure 3 except for the SEG-EAGE salt model (Figure 2b) and with (a) \( f = 5 \text{ Hz} \) and (b) \( f = 20 \text{ Hz} \).](image)
is used to construct the single shot image, and the final image is obtained by superimposing all of the single-shot images together:

\[ I(x, z) = \sum_x I^x(x, z) \]

\[ = \sum_x \text{Re} \left( \sum_\omega u^2(x, z, \omega) u^{*R}(x, z, \omega) \right), \quad (19) \]

where \( I \) is the final image and \( I^x \) is the single-shot image from shot \( S \). The term \( \text{Re} \) stands for the real part of the complex field.

Figure 6a shows the image for the SEG-EAGE salt model obtained using the G-D beamlet propagator. For comparison, we also plot the images migrated using the GSP (Xie et al., 2000) in Figure 6b. All of the main features, including the boundaries of the salt body, sharp edges, steep faults above the salt body, and even parts of the subsalt steep reflectors, are imaged clearly with both the beamlet method and the screen method. The overall image quality of the two methods is comparable except in the interior of the salt body, where the GSP migrated image has a slightly higher noise level.

The intricate structure of the Marmousi model (Figure 7a) produces complicated seismic data, making it difficult to generate images of high quality by many prestack migration methods (Gray et al., 2001). The prestack data set for this model continues to be used as a test bed for migration and velocity-estimation methods. Marmousi images have been obtained using various migration schemes (Geoftrain and Brac, 1993; Gray and May, 1994; Nichols, 1996; Audebert et al., 1997; Han, 1998; Operto et al., 1998; Huang et al., 1999; Hill, 2001; Köhler et al., 2001). We perform prestack G-D beamlet migration for the model and obtain the image shown in Figure 7b. The image from the GSP (Xiaobin Xie, 2002, personal communication) is also plotted in Figure 7c for comparison. Major faults and many thin layers are well imaged by both methods. Again, a lower noise level is achieved with beamlet migration, especially at the upper and middle parts of the image.

Since G-D frame decomposition of wavefields is overcomplete and nonorthogonal, more computations and storage space are required for G-D beamlet-based wavefield propagation and migration. As a result, computation of beamlet migration is several times slower than that of the GSP method. Image comparisons for the two methods, however, indicate that prestack migration can produce images of comparable — even higher — quality for complicated structures. More importantly, the localizations of the wavefield decomposition and propagation in the G-D beamlet domain allow detailed study of the directivity-involved features of local structures. This is particularly desirable for target-oriented prestack migration and imaging.

**TARGET-ORIENTED PRESTACK G-D BEAMLET MIGRATION FOR THE 2D SEG-EAGE SALT MODEL**

In consideration of the huge number of computations required in full prestack migration, especially for 3D surveys, Rietveld et al. (1992) and Rietveld and Berkhout (1994) propose

![Figure 6. Prestack migration images for the SEG-EAGE salt model (a) by G-D beamlet propagator and (b) by generalized screen propagator (GSP) (Xie et al., 2000).](image)

**Table 2. Model, data information, and parameters for prestack beamlet depth migration.**

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Model</th>
<th>Data</th>
<th>G-D beamlet</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N_1 )</td>
<td>( d_x ) (m)</td>
<td>( N_z^1 )</td>
<td>( d_z ) (m)</td>
</tr>
<tr>
<td>SEG-EAGE</td>
<td>645</td>
<td>24.4</td>
<td>150</td>
</tr>
<tr>
<td>Marmousi</td>
<td>369</td>
<td>25</td>
<td>751</td>
</tr>
</tbody>
</table>

- \( N_x, N_z \) are the sample numbers of the model in \( x \) and \( z \) directions with \( d_x \) and \( d_z \) as the sampling interval, respectively. For the prestack data, \( N_x \) is the shot number and \( L_{xy} \) is the trace (receiver) number for each shot. \( N_z \) and \( d_z \) are the length and sampling interval in time of the traces. In G-D beamlet decomposition and propagation, the wavefield at each depth is zero-padded from \( x \)-width of \( L_{xy} \) to \( L_{zy} \). Since the redundancy ratio of G-D frame is kept to be 2, the number of wavelet decomposition coefficients \( N_{x}\) is always equal to \( 2N_x \).
an efficient as well as accurate method to migrate prestack data in a target-oriented way by means of controlled illumination. As a special case of the general approach of areal shot-record migration (Berkhout, 1992), controlled illumination migration is superior to conventional plane-wave stacking (Tanner, 1976; Schultz and Claerbout, 1978) in the sense that control of the source wavefield is put at the target. Another target-illumination-based prestack migration method is proposed by Wu et al. (2002) using partial sources. In that method, both image quality and computation efficiency are improved through selecting only a portion of the sources that contribute illumination energies to the target structures in prestack migration. The principle and procedure of partial source migration apply to any kind of wavefield extrapolation method. In this study, we propose performing target-oriented migration by means of local-angle-domain imaging and partial superposition of common-dip-angle images based on the target structures. In this way, the dual-domain localization property of G-D beamlet decomposition and propagation can be utilized fully, and image quality improvement can be achieved by controlling the image at the target.

Local-angle-domain prestack G-D beamlet migration

In prestack G-D beamlet migration, the incident field (from sources) as well as the scattered field (from receivers) at each extrapolation step (each depth) are obtained as the superposition of contributions from all of the G-D beamlets as expressed in equations 2 or 7. By superimposing only the beamlets with the same local wavenumbers, we get the incident and scattered fields for each individual wavenumber:

\[ u^i(x, z, \hat{\theta}_m, \omega) = \sum_x u^{i}_m (\hat{x} \omega) g_m(x) \]

\[ u^s(x, z, \hat{\theta}_j, \omega) = \sum_x u^{s}_j (\hat{x} \omega) g_j(x), \]  

(20)

where \( u^{i}_m (z, \omega) \) and \( u^{s}_j (z, \omega) \) are beamlet coefficients of the incident field and scattered field at depth \( z \), respectively. The obtained values \( u^i(x, z, \hat{\theta}_m, \omega) \) and \( u^s(x, z, \hat{\theta}_j, \omega) \) are called the directional incident and scattered wavefields at points \( (x, z) \). Since a local wavenumber \( \hat{\theta}_j \) corresponds directly to a propagating angle

\[ \hat{\theta}_j = \sin^{-1} \left( \frac{\hat{x}_j}{\omega} \right) \frac{\hat{y}_j}{\omega} \]  

(21)

where \( \hat{\theta}_j \) is the angle with respect to the vertical. According to equation 21, the directional incident and scattered wavefields can be expressed as \( u^i(x, z, \hat{\theta}_m, \omega) \) and \( u^s(x, z, \hat{\theta}_j, \omega) \) in terms of their propagating angles. Substituting \( u^i(x, z, \hat{\theta}_m, \omega) \) and \( u^s(x, z, \hat{\theta}_j, \omega) \) into the imaging condition 19 in place of the total incident and scattered fields \( u^i(x, z, \omega) \) and \( u^s(x, z, \omega) \), and considering the coordinate transform from \( \hat{x}_m, \hat{\theta}_m \) to \( \hat{x}_j, \hat{\theta}_j \),

\[ \hat{x}_m = k \sin \hat{\theta}_m \]

\[ \hat{d}_m = k \sin \hat{\theta}_m \]

\[ \hat{x}_j = k \sin \hat{\theta}_j \]

\[ \hat{d}_j = k \sin \hat{\theta}_j \]  

(22)

with \( k(x, z) = \omega/\hat{u}(x, z) \) as the wavenumber, the local-angle image matrix is calculated as

\[ I(\hat{\theta}_m, \hat{\theta}_j, x, z) = \cos \hat{\theta}_m \cos \hat{\theta}_j \sum_k k^2(x, z) \]

\[ \times u^2(x, z, \hat{\theta}_m, \omega) u^1(x, z, \hat{\theta}_j, \omega). \]  

(23)

Local-angle-image matrix \( I(\hat{\theta}_m, \hat{\theta}_j, x, z) \) measures the contributions to the final image from different incident-field/scattered-field pairs. We can output images of different incident
angle/scattering angle pairs as an image album. Figure 8 shows the image album for some angle pairs \((\delta_n, \delta_i)\) by common-shot prestack migration for the SEG-EAGE salt model. Here, positive angles represent propagating directions from vertical to the right, and negative angles represent propagating directions from vertical to the left. It is noteworthy that the individual images in the figure highlight the structural features with different orientations. The true dips of the enhanced structures are close to the favored dips predicted based on the planar reflection theory from the propagating directions of the incident-scattered field pairs (marked alongside each panel).

**Target structure-oriented partial dip migration**

Assuming that the local scatterer is a planar reflector, the local-angle-image matrix (2D matrix) can be simplified to different single-variable functions by various stacking methods. Through the coordinate transform from \((\delta_n, \delta_i)\) into \((\delta_s, \delta_r)\),

\[
\begin{align*}
\delta_s &= \frac{\delta_i + \delta_m}{2} \\
\delta_r &= \frac{\delta_i - \delta_m}{2},
\end{align*}
\]

the local-angle-image matrix \(I(\delta_n, \delta_i; x, z)\) is then converted to \(I(\delta_s, \delta_r; x, z)\), with the normal of the reflector \(\delta_m\) and the reflection angle with respect to the normal \(\delta_i\) replacing the incident and scattering angles \(\delta_n\) and \(\delta_i\). On the one hand, images from different normal angles for a fixed common reflection angle can be stacked together to form a common reflection-angle image (CRAI) gather,

\[
I(\delta_r, x, z) = \sum_{\delta_n} I(\delta_n, \delta_r, x, z),
\]

which can be used directly for amplitude variation with angle (AVA) analysis, migration velocity analysis, and estimation of changes in the elastic parameters (Usin et al., 1996; Xu et al., 1998; Brandsberg-Dahl et al., 1999, 2003). On the other hand, because of the mirror reflection of planar interfaces, we can add all of the contributions from different reflection angles, resulting in a set of images as a function of the normal angle of local interfaces:

\[
I(\delta_n, x, z) = \sum_{\delta_r} I(\delta_n, \delta_r, x, z).
\]

This kind of common-angle image gather can be called a common dip-angle image (CDAI) gather since one normal angle corresponds uniquely to a dip angle for a planar interface. The final image of total strength can be obtained by summing up either the CRAI or the CDAI gathers:

\[
I(x, z) = \sum_{\delta_n} I(\delta_n, x, z) = \sum_{\delta_r} I(\delta_r, x, z). \tag{27}
\]

In this study, for the purpose of target-oriented imaging, we consider only the features and applications of CDAI gathers. For local planar reflectors, ideally the CDAI gathers will have peaks at the corresponding real geologic dips. Figure 9 shows the CDAI gathers in the
form of rose diagrams for the SEG-EAGE salt model. At each point a rose diagram is plotted, with each of the 11 petals representing one dip direction. The length of the petal is proportional to the image strength of the corresponding dip. The total image strength for all dip directions is given by the number near each point.

From the figure we can see that most of the image strength comes from the contributions of dip directions in the vicinity of the real dip angle for most of the structures, such as those outside the salt body and the boundary of the salt body. In the subsalt area, however, especially for the three steep faults, not only are the total image strengths much weaker, but the maximum strength dip directions also deviate significantly from the real dips at the upper parts of the faults, coinciding with the relatively poor image quality for these structures as shown in Figure 6. The large discrepancy between the imaged dips of maximum strength and the real dips likely is the consequence of the limited acquisition aperture and the overlying high-velocity salt body that prevents a considerable amount of downward propagated energy from reaching the subsalt area (Wu and Chen, 2002b).

CDAI gathers also can be viewed as an image volume with different dips. Figure 10 gives part of the CDAI image for the subsalt structures. In each dip-angle image, structures with specific directivity features are strengthened while others are relatively weakened. The three subslab steep faults, taken as our target of imaging, are present explicitly in Figures 10b and 10e but can hardly be detected in other panels. To image structures of different directivity features, Brandsberg-Dahl et al. (2005) propose a focusing-in-dip method using cutoff functions in the integration over migration dips to obtain noise-suppressed common scattering-angle image gathers of interesting structures. That technique also provides a tool to estimate local geologic dips and improve image quality.

For complex media such as the subsalt structures of the SEG-EAGE salt model, however, local geologic dips may vary from point to point, which makes it difficult to use the focusing-in-dip procedure to obtain a consistent image for neighboring structures. Considering the real dips of about 32° for the left fault and 44° for the middle and the right faults beneath the salt body, here we superimpose the CDAI with dip angles ranging from 16° to 60° to better image the target faults. In Figure 11, we compare the resultant subsalt image (Figure 11b) with the original migration image (Figure 11a, a zoom of the target area from Figure 6a). Since the contributions of only favorable dip angles are superimposed, the coherent noises, especially those that intersect with the left steep fault, are suppressed markedly. Some other structures, with directivity features much different from those of the steep faults, are also weakened, considerably enhancing the steep faults (Figure 11b). This demonstrates the feasibility of such a structural dip-based CDAI partial summation approach for target-oriented imaging.

In this proposed angle-domain migration and target-oriented imaging scheme, the waveform extrapolation procedure is similar to that of the common-shot prestack G-D beamlet migration. The differences lie only in two aspects: (1) Partial wavefield reconstruction in the local-angle domain is performed instead of full wavefield reconstruction in the space domain to derive the directional incident and scattered fields, and (2) angle-domain contributions are extracted and partially superimposed in imaging at the target. This method clearly does not reduce migration computation costs. However, source field controls (such as those proposed by Rietveld et al. (1992), Rietveld and Berkhout (1994), and Wu et al. (2002)) and simultaneous imaging controls for the target (i.e., combining the target-illumination-based migration method and the present target structure-based partial CDAI summation approach) might provide possible ways to improve image quality with high computation efficiency in target-oriented prestack depth migration.

Figure 10. Common-dip-angle image (CDAI) album for the subsalt structure. (a) Horizontal dip; (b) +30°; (c) +45°; (d) -30°; (e) -45°; (f) total image.
CONCLUSIONS

We have studied and tested beamlet propagation and imaging using Gabor-Daubechies frame propagators for synthetic data sets. The use of local background velocities and local perturbations allows for optimization of the local beamlet propagators and easy handling of the strong lateral velocity variations. The analytical forms of the G-D frame propagator and phase-correction operator are derived based on the one-ray operator decomposition and screen approximation. The prestack data sets of the 2D SEG/EAGE salt model and the Marmousi model are tested by using G-D beamlet propagators in prestack depth migration. The resultant high-quality images demonstrate the capability and potential of the method for imaging complex structures.

Owing to the localization property in both space and direction, beamlet decomposition and extrapolation of wavefields provide localized directional information in each migration step from which local-angle image matrices can be constructed to quantity contributions from different incident-scattered field pairs to the final image. The directional features can also be represented by CDAI gathers through a coordinate transformation and partial summation of the local-angle image matrices. Image quality for the target can be improved by controlled superposition of CDAI based on the directivity feature of target structures. Numerical tests on the 2D SEG/EAGE salt model prestack data show the feasibility and potential of G-D beamlet migration in target-oriented imaging.

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APPENDIX A

GABOR-DAUBECHIES FRAME REPRESENTATION

Since the windowed Fourier transform (WFT) is highly redundant in signal decomposition, and the subsequent WFT reconstruction is very time consuming, many studies have focused on developing sparsely sampled yet accurate decomposition/reconstruction schemes for various signals. The windowed Fourier frame representation, a discretely sampled but complete and stable version of the WFT, was thus developed to overcome the difficulties encountered with the WFT (Daubechies, 1992; Mallat, 1998). In such a representation, an arbitrary function \( f(x) \) can be expanded by a set of windowed Fourier frame functions (vectors) \( \{g_{nm}(x)\} \) that are constructed by space (time) shifting and harmonically modulating a window function \( g(x) \):

\[
f(x) = \sum_{m} \sum_{n} \tilde{f}_{nm} g_{nm}(x) = \sum_{m} \sum_{n} \tilde{f}_{nm} g(x - m\Delta_x)e^{im\Delta_k x}, \tag{A-1}
\]

where \( \Delta_x \) and \( \Delta_k \) are the space (time) and wavenumber (frequency) sampling intervals of the frame vectors and where \( n\Delta_x \) and \( m\Delta_k \) are the corresponding dual-domain loci of the vector \( g_{nm}(x) \). Using the Gaussian window function in equation A-1, Gaber (1996) originally represented signals with sampling intervals satisfying \( \Delta_x \Delta_k = 2\pi \), which was later proven to be the most compact representation of this kind (Daubechies, 1990, 1992). Although the completeness of \( \{g_{nm}(x)\} \) will be guaranteed if the condition \( \Delta_x \Delta_k \leq 2\pi \) is satisfied, Daubechies derived through the frame formalism that the reconstruction under Gaber's critical sampling is unstable and that oversampling \( \Delta_x \Delta_k < 2\pi \) must hold for stabilizing the windowed Fourier frame representation (Daubechies, 1990, 1992). Optimal localized in both space (time) and wavenumber (frequency) domains under the Heisenberg uncertainty principle, the Gaussian window is the most favorable for windowed Fourier analysis of signals (Mallat, 1998).

The windowed Fourier frame with a Gaussian window was named by Daubechies as the Weyl-Heisenberg coherent state frame. Some authors also call it the Gabor frame (Feichtinger and Strohmer, 1998). Here, we call it the Gabor-Daubechies (G-D) frame to emphasize the contributions that Gabor and Daubechies have made to establishing the theory.

In equation A-1 for the windowed Fourier frame representation, \( \tilde{f}_{nm} \) are called the frame decomposition coefficients of the function \( f(x) \). Based upon the frame theorem (Daubechies, 1992; Qian and Chen, 1996; Mallat, 1998), there...
exists a dual-window function $\tilde{g}(x)$ so that $f_{mn}$ can be computed by the regular inner product operation, i.e.,

$$f_{mn} = \langle f, \tilde{g}_{mn} \rangle = \int dx f(x) \tilde{g}_{mn}^{\ast}(x)$$

$$= \int dx f(x) \tilde{g}^{\ast}(x - n \Delta x) e^{-im\Delta x x}, \quad (A-2)$$

where $\ast$ stands for complex conjugate and $\tilde{g}_{mn}(x) = \tilde{g}(x - n \Delta x) e^{im\Delta x x}$ is the dual vector to $g_{mn}(x)$ in the sense that the double summation results in a delta function:

$$\sum_{m} \sum_{n} g_{mn}(x) \tilde{g}_{mn}^{\ast}(x') = \delta(x - x'). \quad (A-3)$$

The set of dual vectors $\{\tilde{g}_{mn}(x)\}$ also constructs a windowed Fourier frame (Daubechies, 1992; Mallat, 1998) that is determined uniquely by the dual-window function $\tilde{g}(x)$.

Compared with the critical-sampling case ($\Delta_x \Delta_t = 2\pi$), frame representation with $\Delta_x \Delta_t = 2\pi$ is redundant and thus nonorthogonal. The redundancy is measured by the redundancy ratio

$$R = \frac{2\pi}{\Delta_x \Delta_t}, \quad (A-4)$$

which is always above the critical value of one. Redundant frame representations result in a nonunique selection of the dual-window function $\tilde{g}(x)$ and, consequently, nonunique dual frames. For such cases, an optimum dual frame should be determined to meet the requirement of the specific purpose. In this work, we select the dual-window function whose shape is the closest to the Gaussian window in the sense of the least-square error to ensure that the dual frame vectors possess good localizations similar to the original Gaussian-windowed frame vectors. The so-defined optimum dual-window function is constructed based on the method proposed by Qian and Chen (1996).

For comparison, the Gaussian window and its dual-window with different redundancy ratios are plotted in Figure A-1, from which we see that the higher the redundancy ratio, the closer the dual-window function is to the Gaussian window function. From a wave propagation point of view, high redundancy in the G-D frame representation of wavefields leads to good localization in both space (time) and wavenumber (frequency). This is a desirable feature for efficient extrapolation of wavefields. However, the computation expense involved in the frame representation and the associated wavefield extrapolation will rise rapidly with increasing redundancy ratio because of the proportionally increased number of frame vectors and decomposition coefficients. This is shown in Table A-1, where the parameters for G-D frame representation of 1D space-domain signals are listed and compared with those for both the WFT and orthogonal representations. As a result, special considerations are required to provide the possibility of a trade-off between the localization property of wavefields and the computation efficiency. This issue is further addressed in the implementation of beamlet-domain wavefield extrapolation in the main text.

In summary, some desirable properties of the redundant G-D frame representation compared with orthogonal representations, especially for wavefield-related studies are (1) optimal space (time)-wavenumber (frequency) localization;

### Table A-1. Parameters used in G-D frame, WFT, and orthogonal representations of 1D space-domain signals.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>WFT</th>
<th>G-D frame</th>
<th>Orthogonal bases</th>
</tr>
</thead>
<tbody>
<tr>
<td>Redundancy ratio, $R$</td>
<td>$L_s$</td>
<td>$&gt;1$</td>
<td>1</td>
</tr>
<tr>
<td>Space-wavenumber sampling, $\Delta_x$</td>
<td>$2\pi/L_s$</td>
<td>$2\pi/R$</td>
<td>$2\pi$</td>
</tr>
<tr>
<td>Number of windows, $N_w$</td>
<td>$L_s$</td>
<td>$L_s/\Delta_x$</td>
<td>$L_s/\Delta_x$</td>
</tr>
<tr>
<td>Number of wavenumbers in each window, $N_t$</td>
<td>$L_s$</td>
<td>$RL_s$</td>
<td>$L_s$</td>
</tr>
<tr>
<td>Number of vectors (decomposition coefficients), $N_{out}$</td>
<td>$L_sL_s$</td>
<td>$RL_sL_s$</td>
<td>$L_s$</td>
</tr>
</tbody>
</table>

WFT — windowed Fourier transform. $L_s$ — length of signal considered. 

Signal sampling interval = 1.

Figure A-1. Gaussian window (dotted lines) and its dual-window (solid lines) with different redundancy ratios: (a) $R = 1.1$; (b) $R = 1.4$; (c) $R = 1.7$; (d) $R = 2$; (e) $R = 4$; (f) $R = 8$. 
APPENDIX B

EXPANSION OF THE SQUARE ROOT OPERATOR $A_\alpha$ AND DECOMPOSITION OF THE THIN-SLAB PROPAGATOR $e^{iA_\alpha \Delta z}$

Considering $k^2(x, z) - k_0^2(x, z)$ as small perturbations, the first-order expansion of the pseudodifferential operator $A_\alpha$ in equation 11 will be

$$A_\alpha \approx \sqrt{\delta^2 + k_0^2(x, z)} + \frac{1}{2} \frac{[k^2(x, z) - k_0^2(x, z)]}{\sqrt{\delta^2 + k_0^2(x, z)}}. \quad (B-1)$$

With the typical small-angle approximation $\delta^2 \ll k_0^2$ and asymptotic phase matching in the forward direction, the second term on the right-hand side can be modified to

$$\frac{[k(x, z) - k_0(\bar{x}, z)][k(x, z) - k_0(\bar{x}, z)]}{2k_0(\bar{x}, z)} \approx k(x, z) - k_0(\bar{x}, z) = \Delta k_\alpha(x, z), \quad (B-2)$$

which is the well-known phase-screen correction. Note that the phase-screen correction matches exactly the traveltime difference in the forward direction. Combining equations B-1 and B-2 results in equation 13 in the main text.

The thin-slab propagator $e^{iA_\alpha \Delta z}$ is accordingly split into two parts: (1) a free propagator $e^{i\sqrt{\delta^2 + k_0^2(x, z)} \Delta z}$, accounting for wavefield propagation in the homogeneous medium with $k_0(\bar{x}, z)$ as the local background wavenumber and (2) a perturbation operator $e^{i\Delta k_\alpha(x, z) \Delta z}$ for the phase-screen correction. After a Fourier transform with respect to $x$, the free propagator takes the form of $e^{i\sqrt{\delta^2 + k_0^2(\bar{x}, z)} \Delta \xi} \hat{\rho}$ (where $\hat{\rho}$ stands for the transverse wavenumber parameter) in the wavenumber domain, in which it can be implemented more efficiently than in the space domain.

APPENDIX C

GABOR-DAUBECHIES BEAMLET-DOMAIN FREE PROPAGATOR

Applying the inverse Fourier transform to equation 15 yields

$$G_0^{mn}(x, z, \Delta z, \omega) = \frac{1}{2\pi} \int d\xi e^{i\xi x} e^{i\Delta k_\alpha \Delta \xi} \hat{\rho}_{mn}(\xi). \quad (C-1)$$

By definition, the G-D beamlet free propagator $P_0^{mn}$ then becomes

$$P_0^{mn}(z, \Delta z, \omega) = \langle G_0^{mn}, \hat{g}_{jl} \rangle$$

$$= \frac{1}{2\pi} \int dx \hat{g}_{jl}^*(x) \int d\xi e^{i\xi x} e^{i\Delta k_\alpha \Delta \xi} \hat{\rho}_{mn}(\xi)$$

$$= \frac{1}{2\pi} \int d\xi e^{i\Delta k_\alpha \Delta \xi} \hat{\rho}_{mn}(\xi) \int dx \hat{g}_{jl}(x) e^{i\xi x}$$

$$= \frac{1}{2\pi} \int d\xi e^{i\Delta k_\alpha \Delta \xi} \hat{\rho}_{mn}(\xi) \hat{g}_{jl}(\xi) e^{i\xi x}. \quad (C-2)$$

where

$$\hat{g}_{jl}(\xi) = \int dx \hat{g}_{jl}(x) e^{i\xi x} = \int dx \hat{g}(x - \bar{x}_j) e^{i\xi(x - \bar{x}_j)},$$

$$= \hat{g}^*(\xi - \xi_j) e^{i\xi(x - \bar{x}_j)}, \quad (C-3)$$

and

$$\hat{\rho}_{mn}(\xi) = \int dx \rho(x - \bar{x}_i) e^{i\xi n \psi x} e^{-i\xi x}$$

$$= \hat{\rho}(\xi - \xi_m) e^{-i\xi m \psi x}. \quad (C-4)$$

Substituting equations C-3 and C-4 into the last line of equation C-2, we obtain the final expression for the free propagator (equation 16 in the main text).

APPENDIX D

LATERAL-PERTURBATION PHASE CORRECTION

From equations 4, 14, and 16 and following the derivation of equation 6 in the main text, we have

$$u(x, z + \Delta z, \omega) = \sum_m \sum_n u_{mn}(z, \omega)$$

$$\times e^{i\Delta \Delta k(x, z, \omega) \Delta z} G_0^{mn}(x, z, \Delta z, \omega)$$

$$= \sum_m \sum_n u_{mn}(z, \omega) e^{i\Delta \Delta k(x, z, \alpha) \Delta z}$$

$$\times \sum_j \sum_l \sum_m P_0^{mn}(z, \Delta z, \omega) g_{jl}(x)$$

$$= \sum_j \sum_l \left[ \sum_m \sum_n P_0^{mn}(z, \Delta z, \omega) u_{mn}(z, \omega) \right] g_{jl}(x). \quad (D-1)$$

In contrast to the free propagation of wavefields, which is implemented in the beamlet domain, the perturbation-related phase correction is carried out in the local space-beamlet mixed domain. The term $u_{mn}(z, \omega) e^{i\Delta \Delta k(x, z, \omega) \Delta z}$ in equation D-1 can be approximated by connecting the lateral perturbations.
within individual windows:

\[
\begin{align*}
\Delta u_{mn}(z, \omega) & = \int d\tilde{x} e^{i\tilde{x}y} \frac{\Delta \hat{u} (z, \tilde{x}, \omega)}{2} \int d\tilde{x} ' u (x', \tilde{x}, z, \omega) e^{i(\tilde{x} ' - x) y} e^{i(\tilde{x} ' - \tilde{x})} \Delta \hat{u} (z, \tilde{x}, \omega) \Delta \hat{u} (z, \tilde{x}, \omega) \\
& \approx \int d\tilde{x} u(x', \tilde{x}, z, \omega) e^{i\Delta \hat{u} (z, \tilde{x}, \omega) \Delta \hat{u} (z, \tilde{x}, \omega)} g(x'-x) e^{-i(\tilde{x} ' - \tilde{x})} \\
& = u_{mn}(x, \omega).
\end{align*}
\]

(D-2)

The phase-corrected beamlet coefficients \( u_{mn} \) are used to account for the effect of lateral heterogeneity and are then applied to beamlet-domain free propagation, as expressed in equation 18 in the main text.

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Directional Illumination Analysis Using Beamlet Decomposition and Propagation

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Abstract

We evaluate directional illumination and acquisition-aperture efficacy through wave-theory based beamlet decomposition of the wavefield. Beamlet decomposition (wavelet transform along spatial axis) provides localizations in both space and direction of a wave field. We introduce the image conditions in beamlet domain and local angle domain, and then define the local image matrix (LIM). We calculate the directional illumination (DI) in the image space for a given source or a group of sources by decomposing Green’s functions into local angle-domain at image points. Acquisition-aperture efficacy (AAE) matrix and acquisition dip-response (ADR) vector can be defined to quantify the efficacy of an acquisition configuration for a given subsurface point. As numerical examples, we calculate the directional illumination maps and acquisition dip-response maps for high- and low-velocity lens models and for the SEG-EAGE 2D salt model. We further investigate the influences of acquisition geometry and overlaying structures on the quality of prestack depth migration image for the subsalt area of the SEG-EAGE model. We find that the acquisition dip-response maps for different dip-angles have good correlation with the image qualities of the corresponding reflectors. Directional illumination analysis can be used in the aperture correction for image amplitude in local angle-domain for wave-theory based migration methods.

Introduction

Although seismic imaging techniques, especially prestack depth migrations, have been improved significantly in providing reliable high-resolution seismic images for complex structures, there remains a need to better understand various factors affecting image quality, such as acquisition geometry, various approximations involved in migration procedures, and the influences of overlaying velocity structures above target areas. Illumination analysis in the target areas is a powerful tool to study the influences of acquisition aperture and overlaying structures. In the past, techniques for illumination studies are often based on ray tracing modeling (Berkhout, 1997; Muerdter et al., 2001a; 2001b; 2001c; Schneider, 1999; Bear et al., 2000). Although ray tracing is convenient and its results can be easily sorted into CMP (common mid-point) gathers, common offset gathers, or common reflection angle gathers to understand illumination attributes, the resulted illumination maps may bear large errors in complex areas due to the high frequency approximation and singularity problems of ray theory for complex regions (Hoffmann, 2001). On the other hand, finite-difference modeling for illumination analysis avoids such problems. The issue with finite-difference methods is that a space domain solution of the wave equation can provide only the total illumination at any point. The ray method seemingly can provide angle-dependent illumination at any point. In fact, this “over-precise” directional illumination map does not reflect the real behavior of the wave field since it violates the Heisenberg uncertainty
principle: the position and direction of a wave field cannot be accurately specified at the same
time. In order to have reliable directional illumination, we need to have a wave-theory based
method which possesses both space and direction localizations with their widths satisfying the
uncertainty principle.

Recently, a wave theory-based beamlet wave propagation and migration method has been
developed (Wu et al., 2000; Wu & Chen, 2001). Instead of global FT (Fourier transform), Wu et
al. (2000) applied efficient decomposition schemes using Gabor-Daubechies frame (G-D frame)
or local cosine bases to the wave field and derived the corresponding propagators in the beamlet
domain (Wu et al., 2000; Wu and Chen, 2001; 2002a). The wave field decomposition element
(atom) in beamlet transform is a windowed harmonics along spatial axes, and therefore it is
termed “beamlet” (wavelet along spatial axis). In case of the G-D frame, translated and
modulated Gaussian window functions are used to construct the frame atoms for beamlet
decomposition. The beamlet decomposition provides localizations in both space and direction
(local wavenumber) of the wave field, making it natural for analyzing directional illumination
distributions. Beamlet domain propagators are also more flexible and have better wide-angle
performances than traditional global propagators.

In this study, we define and evaluate illuminations distribution and acquisition dip-
responses for subsurface reflectors for a given acquisition geometry and overlaying structures
through beamlet wave field decomposition and propagation. First, local plane waves are defined
based on the G-D frame beamlet decomposition. Then a spatial distribution of directional
illumination (DI) is derived through the decomposition of Green’s function into the beamlet
domain (or local angle-domain). In order to calculate the acquisition aperture efficacy (AAE)
that includes the effects of both source and receiver apertures, we introduce the image condition
in beamlet domain and mixed domain (local phase space), and define the local image matrix as
the migrated images in local angle domain (as a function of incident-receiving angle pairs). A
local AAE matrix measures the system response in local angle domain at each point. To simplify
the presentation, the AAE matrix is reduced to a local acquisition dip response (ADR) that
measures the total system response to local reflectors with different dip-angles. A lens model
and the SEG-EAGE 2D salt model are used as examples to demonstrate the feasibilities of the
approach. The influences of various factors on the final prestack image quality through DI and
ADR analyses are investigated for the SEG-EAGE salt model.

**Beamlet Decomposition of Wave Field**

Wave fields in the frequency-space domain can be represented as $u(x_T, z)$ for a given
frequency $\omega$, where $x_T$ is the horizontal position vector $x_T = (x, y)$ in the 3D case, and $x_T = x$ in
the 2D case, and $z$ is the depth. For the sake of simplicity, we omit the parameter $\omega$ in the wave
field representation. Equivalently, it can be represented in the frequency-wavenumber domain as
$u(k_T, z)$, where $k_T = (k_x, k_y)$ is the horizontal or transverse wavenumber vector. However, these
two domains cannot coexist simultaneously. In space domain the space localization is perfect.
That means you can specify precisely the phase and amplitude of a wave field at any point; but
you can not specify the propagation direction at all. On the other hand, in wavenumber domain
you can have perfect direction localization, but no space localization at all. This is one of the
most important differences of the traditional wave field representation (either space-domain or
wavenumber-domain) from the ray representation (asymptotic approximations).
Recent progress in beamlet decomposition of wave field (Steinberg, 1993; Wu et al., 2000; Wu and Chen, 2001, 2002a, b; Xie and Wu, 2002) provides a basis for localizing the wave field in both space and direction simultaneously. Beamlet transform uses translated windows for spatial localization and harmonic modulations for directional localization. The sizes of spatial and directional localizations cannot be arbitrarily small simultaneously and must satisfy the Heisenberg uncertainty principle. This is an important difference between the wave theory in beamlet-domain and the ray theory, similar to the difference between the quantum mechanics and the classical mechanics.

In this study we will use Gabor decomposition (Gabor-Daubechies transform). The Gabor atom (decomposition element) is a Gaussian windowed exponential harmonics

$$g_{mn}(x) = e^{im\Delta_x x} g(x - n\Delta_x), \quad (1)$$

where $g(x)$ is a window function, $\Delta_x$ and $\Delta_{\xi}$ are the space and wavenumber sampling intervals respectively. $n\Delta_x$ and $m\Delta_{\xi}$ are the corresponding space-wavenumber domain loci of $g_{mn}(x)$. This decomposition atom is the same as the windowed Fourier transform. However, in order to have efficient decomposition and reconstruction, we use the Gabor-Daubechies frame theory, which is also called Gabor frame (Feichtinger and Strohmer, 1998) or Weyl-Heisenberg coherent state frame (Daubechies, 1990) because of its relation with the Weyl-Heisenberg group in quantum field theory (Klauder and Skagerstam, 1985; Foster and Huang, 1991). We put the Gabor-Daubechies frame theory to the Appendix A, and give only the necessary definition and physical explanation in this section.

A G-D frame is a type of windowed Fourier frame (Daubechies, 1992) using the Gaussian window function. Frame decomposition is not orthogonal and therefore has redundancy in the representation. Gabor (1946) originally proposed a decomposition using the critical sampling $\Delta_x\Delta_{\xi} = 2\pi$ in the time-frequency domain. Daubechies (1990) has proved that the reconstruction using critical sampling is unstable, and for stable reconstruction over sampling ($\Delta_x\Delta_{\xi} < 2\pi$) must hold, where $\Delta_x\Delta_{\xi}$ measures the size of windowed Fourier atoms (latticed coherent states). The necessary and sufficient conditions for the stable reconstruction have been derived based on the frame theory (Daubechies, 1990; 1992). In Wu et al. (2000), this Gaussian windowed Fourier frame is called the Gabor-Daubechies (G-D) frame. For 2D cases, beamlet decomposition of a wave field at depth $z$ using the G-D frame can be expressed as (Wu et al., 2000; Wu and Chen, 2001):

$$u(x, z, \omega) = \sum_m \sum_n \langle u, \tilde{g}_{mn} \rangle g_{mn}(x) \quad (2)$$

where $\tilde{u}_z(\vec{x}_n, \vec{\xi}_m, \omega)$ are the beamlet coefficients, $\omega$ is the circular frequency, $g_{mn}$ and $\tilde{g}_{mn}$ are the G-D frame atoms and dual frame atoms respectively:

$$g_{mn}(x) = e^{im\xi n} g(x - \vec{x}_n), \quad \tilde{g}_{mn}(x) = e^{im\xi n} \tilde{g}(x - \vec{x}_n), \quad (3)$$

where $\vec{x}_n = n\Delta_x, \quad \vec{\xi}_m = m\Delta_{\xi}$ with $\Delta_x\Delta_{\xi} < 2\pi$ are the $n$th window location and the $m$th local wavenumber position respectively. $g(x)$ is a Gaussian window function with $\tilde{g}(x)$ as its dual.
window function. The dual window function can be calculated by pseudo-inversion of the original window function (Mallat, 1998; Qian and Chen, 1996; Wu and Chen, 2001). For wave field propagation in beamlet domain, we decompose the wave field with the G-D frame such that its dual window function is very close to the Gaussian window function. This is the case of tight frame (see Appendix A or Daubechies, 1992). We see that for wave field decompositions, each beamlet (in this case a G-D frame atom) is a \textit{windowed plane wave} that has both space localization ($\bar{x}_n$) and direction localization ($\bar{\xi}_m$). Due to the uncertainty principle $\Delta x \Delta \xi < 2\pi$, the local parameters in beamlet domain $\bar{x}_n$ & $\bar{\xi}_m$ are different from those in the space-wavenumber domain $x$ & $\xi$. The beamlet position is only specified as a local window centered at $\bar{x}$, and $\xi$ specifies only the lobe direction of the beamlet centered at $\bar{\xi}$ in the wavenumber domain. The Gabor beamlet (Gaussian beamlet) has a smooth lobe without side lobes. The width of the lobe is inversely proportional to the width of the spatial window. Beamlets can be propagated by propagators in beamlet domain and then form images by applying the imaging condition in either the beamlet domain or the space domain (Wu et al., 2000; Wu and Chen, 2001). At each step, the wave field in the space domain can be reconstructed by summing up the contributions from the beamlets (inverse beamlet transform):

\begin{equation}
\sum_{j} e^{i\bar{\xi}_j x} \sum_{l} g(x - \bar{x}_l) \hat{u}_z(\bar{x}_l, \bar{\xi}_j, \omega).
\end{equation}

Although we propagate the wavefield using beamlet propagator, the wave propagation can be implemented by any wave-theory based propagators that can faithfully calculate amplitude information in the local angle-domain. It can be seen from equation (4) that at each space location the field can be recovered by superposing the contributions of all the windowed plane waves (beamlets) from all the neighboring windows. Due to the nature of Gaussian windows, for each $x$, the field is mainly controlled by the beamlets in a few neighboring windows. We can also have a partial reconstruction (mixed domain wave field in local phase space):

\begin{equation}
\sum_{j} e^{i\bar{\xi}_j x} \sum_{l} g(x - \bar{x}_l) \hat{u}_z(\bar{x}_l, \bar{\xi}_j, \omega).
\end{equation}

We call $u(x,z,\bar{\xi}_j, \omega)$ a \textit{local plane wave}, which is a superposition (weighted average) of windowed plane waves of the same local wavenumber (beamlets with the same lobe direction and lobe width) from neighboring windows. Therefore the local plane wave for location $x$ is an average beamlet over its neighboring windows. For the local plane wave of local wavenumber $\bar{\xi}_j$, the corresponding propagating angle is

\begin{equation}
\bar{\theta}_j = \sin^{-1}(\bar{\xi}_j \cdot v(x,z)/\omega),
\end{equation}

where $\bar{\theta}_j$ is the local incident angle with respect to the vertical, and $v(x,z)$ is the wave velocity at $(x, z)$. Note that a local plane wave is not a plane wave in the normal sense. It is a beamlet with a Gaussian lobe whose width is inversely proportional to the window width in space-domain. Each $\bar{\xi}_j$ corresponds to an angle of the lobe direction. For other directions between the specified lobe directions, interpolation between neighboring lobes can be used.
Directional Illumination Map

A definition of directional illumination (DI) can be illustrated by Figure 1. For the single-source case, different space points receive illumination energy of different directions from the source; for multi-sources, the DI intensity at each point is the sum of DI energy from all the sources.

The computation procedure of DI maps is as follows. For a single source, we put a unit-strength source on the surface and propagate the field to the image space. Through partial reconstruction of the beamlet domain fields, we can get the local plane waves (average beamlets) with $G(x,z,\bar{\xi}_j,x_s)$ or $G(x,z,\bar{\theta}_j,x_s)$, which is the Green’s function in local angle domain, where $x_s$ is the source location on the surface. For the illumination problem, we are only concerned with the intensity or amplitude. For a single frequency $\omega_0$ (such as the dominant frequency of a source time function), the DI-map of a single source is defined as

$$ D_a(x,z,\bar{\theta}_j;x_s,\omega_0) = \left| G(x,z,\bar{\theta}_j;x_s,\omega_0) \right|, $$ (7)

where $a$ is the window width of the G-D frame used in wave field decomposition, representing the range of average for deriving local plane waves. We can also calculate and display the average illumination within a small frequency band around $\omega_0$,

$$ D_a(x,z,\bar{\theta}_j;x_s,\omega_0 \pm \Delta \omega) = \left[ \sum_{\omega_0 - \Delta \omega}^{\omega_0 + \Delta \omega} \left| G(x,z,\bar{\theta}_j;x_s,\omega) \right|^2 \right]^{1/2} = \left[ \sum_{\omega_0 - \Delta \omega}^{\omega_0 + \Delta \omega} \left( x,z,\bar{\theta}_j;x_s,\omega \right) \right]^{1/2}. $$ (8)

In the same way, the DI-map can be calculated by summing up the energy of local plane waves over all the frequencies.

By summing up the DI-maps of individual sources, the DI-map of a group of sources or all the sources in an acquisition system can be calculated as well. Taking point sources (in shot domain) as an example:

$$ D_a(x,z,\bar{\theta}_j) = \left[ \sum_{S=1}^{N_s} D_a^{(2)}(x,z,\bar{\theta}_j;x S) \right]^{1/2}, $$ (9)

where $N_s$ is the number of sources. For group sources of other types, such as synthetic beam sources or plane sources, or generally “areal shots” (Berkhout, 1992), the summation procedure is similar. The frequency variable has been dropped in equation (9), implying that the calculated multi-source DI-map has the same frequency contents as that of the single-source DI-maps used.

To demonstrate the procedure of DI mapping, we consider a lens model consisting of a homogeneous lens embedded in a homogeneous background. The velocity of the background medium $v_0$ is 2000m/s. Two different velocities are chosen for the lens: $v_{\text{lens}} = 4000$ m/s and $v_{\text{lens}} = 1000$ m/s corresponding to velocity perturbations of +100% and -50%, respectively. For both cases, an acquisition system consisting of 257 shots with 176 left-hand-side receivers for each shot, i.e. $N_S = 257$ and $N_R = 176$, is used to calculate the directional illuminations and the acquisition system response. A Ricker wavelet with the dominant frequency $f_0 = 15$ Hz is used as the source time function.
In Figure 2, the directional illumination gathers of the acquisition system using the dominant frequency are shown in the form of “rose diagram”. Figure 2a is for +100% velocity contrast, and 2b, for -50%. At each point in these maps, 11 arrows (petals) are plotted, each of which represents one illumination direction. The number near each point gives the total illumination strength at that point. The lengths of the arrows indicate the relative strengths of illumination for different directions. As expected, the DI-maps show clearly the directivity features of acquisition illumination. For both velocity perturbation cases, the total illumination strength decreases with depth. However, the intensity distributions and directivity features inside the lens body and at the sub-lens area are different. For the high velocity lens, directional illuminations show the typical features of wave field defocusing; while for the case of low velocity lens, focusing features can be seen clearly. At the sub-lens area, especially near the lens boundary, the two DI-maps have different appearances due to their different energy transmission patterns.

The other form of presenting the DI-map is the DI-map album which is the collection of DI intensity maps \( D_a(x, z, \vartheta, \omega_0) \) for different incident angles \( \vartheta \), as shown in Figure 3a and 3b. We can see that due to the high velocity contrasts, the DI distributions in both cases look highly non-uniform. As a result, some “shadow zones” appear within and under the lens body in the DI albums of different directions, including the case of vertical incidence. However, the distribution patterns of the shadow zones are very different for the two velocity contrasts. The defocusing in the high-velocity lens case (+100% contrast) and the focusing of the low-velocity lens (-50% contrast) can be seen clearly in these figures.

To investigate the frequency dependence of the DI distributions, first we consider the frequency spectrum of the source function, the Ricker wavelet \( f_0 = 15 \text{Hz} \). The half energy points of such a spectrum corresponds to \( f_1 = 9.3 \text{Hz} \) and \( f_2 = 21.6 \text{Hz} \). The DI albums of the acquisition system for \( f_1 \) and \( f_2 \) are calculated and compared with the case of dominant frequency \( f_0 \). We see no significant differences between these maps and the maps for the dominant frequency. We conclude that DI distributions are not sensitive to frequency at least within the frequency range containing significant energy.

**Image Conditions in Beamlet Domain and Local Image Matrix**

By performing beamlet wave field decomposition and propagation, and applying the widely used image condition in space domain introduced by Claerbout (1971) to the decomposed wave fields or partially reconstructed wave fields, the image conditions in beamlet domain and in local angle domain can be obtained accordingly.

Figure 4 shows the concepts of wave field decomposition and the derivation of a local image matrix. For each point source, the forward-propagated wave field can be decomposed locally at the image point \((x, z)\),

\[
 u^s(x, z, \omega) = \sum_j \sum_l \hat{u}^s_j(\tilde{x}_j, \tilde{\zeta}_j, \omega) g^s_{jl}(x),
\]  

and the received scattered wave field at each receiver can be back-propagated to the image space and decomposed locally at the same image point,

\[
 u^r_s(x, z, \omega) = \sum_p \sum_q \hat{u}^r_s(q, \tilde{\xi}_q, \omega) g^r_{pq}(x).
\]
Here the superscripts $S$ and $RS$ refer to the point source and the point receiver (for source $S$) located on the surface, respectively. Taking the cross-correlation of $u^S(x, z, \omega)$ and $u^{RS}(x, z, \omega)$, and summing up the contributions from all the source-receiver pairs, we obtain the image in frequency domain. Using equations (10) and (11), and writing explicitly $g_{jl}$ and $g_{pq}$ (equation 3), we obtain

$$I(x, z, \omega) = \sum_S u^S(x, z, \omega) \left[ \sum_{RS} u^{RS}(x, z, \omega) \right]$$

$$= \sum_j \sum_p e^{i(\xi_j - \xi_p)k} \sum_l \sum_q \left[ \sum_S \hat{u}_z^S(\xi_l, \xi_j, \omega) \left( \sum_{RS} \hat{u}_z^{RS}(\xi_q, \xi_p, \omega) \right) \right] g(x - \xi_j) g(x - \xi_q), \quad (12)$$

$$= \sum_j \sum_p e^{i(\xi_j - \xi_p)k} \sum_l \sum_q M_{jlpq} g(x - \xi_j) g(x - \xi_q)$$

with

$$M_{jlpq}(\xi_l, \xi_j, \xi_q, \xi_p) = \sum_S \hat{u}_z^S(\xi_l, \xi_j, \omega) \sum_{RS} \hat{u}_z^{RS}(\xi_q, \xi_p, \omega). \quad (13)$$

We call $M_{jlpq}$ the **image matrix in the beamlet domain**. It is the image produced by the incident windowed plane wave in $l$th window with $j$th wavenumber and the scattered windowed plane wave in $q$th window with $p$th wavenumber. Summing up the contributions from all beamlets with the same local wavenumber in the neighboring windows, we get the **image matrix in local angle-domain (the local image matrix)** for a specific frequency $\omega$.

$$L_a(\xi_j, \xi_p, x, z, \omega) = \sum_l \sum_q M_{jlpq} g(x - \xi_j) g(x - \xi_q) e^{i(\xi_j - \xi_p)k}$$

$$= \sum_S \left[ e^{i\xi_jk} \sum_l \hat{u}_z^S(\xi_l, \xi_j, \omega) g(x - \xi_j) \right] \left[ \sum_{RS} e^{-i\xi_pk} \sum_q \hat{u}_z^{RS}(\xi_q, \xi_p, \omega) g(x - \xi_q) \right]. \quad (14)$$

$$W^S_a(x, \xi_j, z, \omega) = \sum_{RS} W^{RS}_a(x, \xi_p, z, \omega)$$

$W^S_a(x, \xi_j, z, \omega)$ and $W^{RS}_a(x, \xi_p, z, \omega)$ in the above formula are **local incident plane wave** (incident beamlet) and **local scattered plane wave** (scattered beamlet) respectively. The window width is $a$.

We can express the local image matrix as a function of local incident and receiving angles by the coordinate transform from $\xi_j, \xi_p$ to $\theta_j, \theta_p$, where $\theta$’s are the angles with respect to the vertical ($z$) axis (See Figure 6):

$$\xi_j = \theta_j = k_0 \sin \theta_j, \quad |d\theta_j| = |d\theta| |k_0 \cos \theta_j|$$

$$\xi_p = \theta_p = k_0 \sin \theta_p, \quad |d\theta_p| = |d\theta| |k_0 \cos \theta_p|,$$

with $k_0 = \omega/\nu(x, z)$. $\theta_j$ and $\theta_p$ are the local incident and receiving angles respectively. Therefore
where \( k_0^2 (\cos \bar{\theta}_i \cos \bar{\theta}_g) \) serves as the Jacobian of the coordinate transform, similar to the case of scattering tomography (linearized inversion) (Wu and Toksöz, 1987; Miller et al., 1987; Wu et al., 1994). In this case, the effect of the heterogeneous overburden has been corrected by the forward propagation (downward continuation) and the inversion is a local inversion in the homogeneous background.

Stacking the image matrices of all the frequencies with incident angle \( \bar{\theta}_i \) and receiving angle \( \bar{\theta}_g \) (image condition) results in images in the local angle domain,

\[
I(\bar{\theta}_i, \bar{\theta}_g, x, z) = \text{Re}\int d\omega L_0(\bar{\theta}_i, \bar{\theta}_g, x, z, \omega) .
\]

The summation over frequency implies that the local incident and scattered plane waves meet at the image point \((x, z)\) at \(t = 0\). \(I(\bar{\theta}_i, \bar{\theta}_g, x, z)\) is called the local angle-domain image matrix.

The final image of total strength in the space domain can be obtained from the local image matrix by summing up the contributions of all scattering experiments \((\bar{\theta}_i, \bar{\theta}_g)\):

\[
I(x, z) = \sum_{\bar{\theta}_i} \sum_{\bar{\theta}_g} I(\bar{\theta}_i, \bar{\theta}_g, x, z) .
\]

The local angle coordinates in incident (or source)-receiving angle pairs \((\bar{\theta}_i, \bar{\theta}_g)\) can be transformed to normal-reflection angle pairs \((\bar{\theta}_n, \bar{\theta}_r)\). The definition of the wavenumber vectors and the related angles are shown in Figures 5 and 6a. A wavenumber vector \(\vec{k}\) is defined as \(\vec{k} = (\vec{\xi}, \zeta)\), where \(\vec{\xi}\) is the horizontal wavenumber and \(\zeta\) is the vertical wavenumber. In the 2D case, \(\vec{\xi} = \xi\). As can be seen from Figure 5, the exchange wavenumber vector \(\vec{k}_n\) is defined as

\[
\vec{k}_n = \vec{k}_g - \vec{k}_i = \vec{k}_g + \vec{k}_s ,
\]

where \(\vec{k}_g\) and \(\vec{k}_i\) are the receiving and incident wavenumber vectors, respectively (Morse and Feshbach, 1953; Wu and Toksöz, 1987; Sato and Fehler, 1998). The \(\vec{k}_i = -\vec{k}_s\) is the source wavenumber vector. In the case of common-mode scattering (P-P or S-S), \(\vec{k}_g\) and \(\vec{k}_s\) have the same magnitude, but with different directions. However, in the case of converted waves, they can have different magnitudes too. \(\vec{k}_n\) represents the vectorial spectral component of a local heterogeneity that can be revealed by a local scattering (single frequency) experiment (see, e.g. Wu and Toksöz, 1987). In the case of local planar reflectors, the direction of \(\vec{k}_n\) (defined by the unit vector \(\hat{n}\)) coincides with the normal direction of the reflector that can be detected by this \((\vec{k}_g, \vec{k}_s)\) pair. For the case of acoustic waves (P-P scattering), we have the angle coordinate transform from \((\bar{\theta}_i, \bar{\theta}_g)\) to \((\bar{\theta}_n, \bar{\theta}_r)\):
\[ \bar{\theta}_n = \left( \bar{\theta}_g + \bar{\theta}_i \right) / 2 \]
\[ \bar{\theta}_r = \left( \bar{\theta}_g - \bar{\theta}_i \right) / 2, \]  
where \( \bar{\theta}_n \) is the normal angle of the dipping reflector and \( \bar{\theta}_r \) is the reflection angle with respect to the normal. Because of the mirror reflection of planar interfaces, we can sum up all the responses for different reflection-angles for a common normal-angle, resulting in
\[ I(\bar{\theta}_n, x, z) = \sum_{\bar{\theta}_r} I(\bar{\theta}_n, \bar{\theta}_r, x, z). \]  
This kind of reflector-dip image gathers is one type of the CAI (common-angle image) gathers, and can be called CDAI (common dip-angle image) gathers. For local planar reflectors, the CDAI gathers will have peaks at the corresponding real dip-angles.

For the purpose of local AVA (amplitude versus angle) analysis we can sum up all the elements of different dip-angles for a common reflection-angle,
\[ I(\bar{\theta}_r, x, z) = \sum_{\bar{\theta}_n} I(\bar{\theta}_n, \bar{\theta}_r, x, z), \]  
resulting in CRAI (common reflection-angle image) gathers. More generally we can also get CIAI (common incident-angle image) and CSAI (common scattering-angle image) gathers from local image matrices for different purposes.

**Local Scattering Matrix And Local Image Matrix**

The above defined local image matrix is closely related to the local scattering matrix which we will define in the following. For a scatterer (heterogeneity) located in a homogeneous background, the scattering amplitude is defined as the complex amplitude \( A(\theta_i, \theta_g) \) of the scattered field as a function of scattering angle \( \theta_{sc} = \theta_g - \theta_i \) with respect to a unit incident plane wave at \( \theta_i \) (Ishimaru, 1978). The squared amplitude is referred to as scattered energy by a scatterer. Scattering coefficient \( E_s(\theta_i, \theta_g, x, z) \) for a continuous, heterogeneous medium is defined as the scattered energy within a unit solid angle around \( \theta_g \) by a unit volume of heterogeneity centered at \( (x, z) \) with respect to the incident plane wave of a unit energy flux (Aki and Richard, 1980; Wu 1985). The corresponding definition for discrete scatterers is the scattering cross section \( \sigma_s(\theta_{sc}, x, z) \) (Ishimaru, 1978), and \( E_s = n \sigma_s \), where \( n \) is the number density (number of scatterers within a unit volume). Assume we can conduct scattering experiments surrounding the local heterogeneity at \( (x, z) \) with a series of local incident plane waves at different angles, we can then define the local scattering matrix \( S(\bar{\theta}_i, \bar{\theta}_g ; x, z) \) as the matrix of scattering coefficients for incident-scattering angle pairs \( (\bar{\theta}_i, \bar{\theta}_g) \), which quantifies the angle distribution of scattered energy for different incident angles (see the conceptual sketch in Figure 4). The local scattering matrix (LSM) is the intrinsic property of the scattering medium and is independent of the acquisition system and free from propagation effects. If the heterogeneous medium is a statistically uniform or slowly varying random medium, then the LSM defines the average scattering property of the random medium. For deterministic problems, the LSM
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contains information of the local structure and the elastic properties revealed by the scattering experiments at location \((x, z)\). For acoustic media, it is mainly controlled by the local structure and the impedance contrast. In the point-scatterer case, the LSM is a uniformly distributed, full matrix, indicating its isotropic scattering property; while for a local planar reflector with dip-angle \(\theta_d\), the LSM degenerates into a line segment perpendicular to the matrix diagonal as illustrated in Figure 6b. On the left (Figure 6a) shows the angle relations of scattering experiments; on the right (Figure 6b), the structure of the LSM. Without aperture limitation \((360^\circ\) experiments\) the line segment of the LSM has the same length but situated at different locations along the diagonal for different dip-angles (light lines in Figure 6b). This can be understood from the conservation of the total scattered energy. In this case, the dip-angle of the reflector can be easily determined from the structure of the LSM. However, in surface reflection measurements, data are only available on the surface. Even if the source (shot) and receiver (geophone) both have unlimited apertures on the surface, the line segment of the LSM will have different length for different dip-angles as shown in Figure 6b (bold lines). Horizontal reflector has the longest length and vertical reflector has zero length. For other types of structures, such as curved interface, sharp edge, etc., the corresponding LSM will have different shapes and values.

Local image matrix (LIM) is the distorted LSM due to the aperture limitation of surface experiments (data acquisition) combined with the propagation effects of overburdens (see Figure 4). As an extreme case, we consider zero-offset experiments where the receiver aperture is a single point coinciding with the shot point. In this case the LIM for a planar reflector becomes a single point, since scattered energy can be received for only one pair of angles \((\bar{\theta}_s, \bar{\theta}_r)\). For limited apertures, the LIM’s of planar reflectors will look like something between the two extremes of full aperture and point aperture. As shown on the right panel of Figure 7, the LIM’s of a reflector with 30° dip situated at different depths have different lengths of line segment for limited acquisition aperture (total 201 shots with 176 left-hand receivers per shot). For a point scatterer, the LIM’s with this acquisition system are shown on the left panel of Figure 7. The propagation effects of the overburdens can change the effective local acquisition aperture dramatically so that the LIM is further deviated from the LSM of the target. The purpose of high fidelity imaging is to recover or partly recover the true local scattering matrix from the local image matrix by removing the propagation and aperture effects.

Local Acquisition Aperture Efficacy Matrix and Acquisition Dip-Response Vector

In order to evaluate the effects of the acquisition geometry for a specific target area, including the aperture and propagation effects, we calculate the Green’s function in local angle domain at image points for all the source and receiver locations on the surface for the whole acquisition configuration. Similar to the procedure of DI mapping, we neglect the detailed wave interference pattern and consider only the energy distribution of the acquisition configuration. We define

\[
E(\bar{x}_j, \bar{\xi}_p, x, z, \omega) = \left[ \sum_s |G(x, z, \bar{x}_j; x_s, \omega)|^2 \sum_{R_k} |G(x, z, \bar{\xi}_p; x_{R_k}, \omega)|^2 \right]^{1/2}
\]  

(22)
as the local Acquisition Aperture Efficacy matrix (AAE matrix), where

\[
G(x, z, \vec{x}_j; x_s, \omega) = e^{j \vec{\xi} \cdot x} \sum_j \hat{G}_s(\vec{x}_j, \vec{x}_s; x_s, \omega) g(x - \vec{x}_i)
\]

and

\[
G(x, z, \vec{x}_p; x_s, \omega) = e^{j \vec{\xi} \cdot x} \sum_q \hat{G}_s(\vec{x}_q, \vec{x}_p; x_s, \omega) g(x - \vec{x}_q)
\]

are the local angle-domain decomposition of the impulse responses (Green’s functions); \( \hat{G}_s(\vec{x}_j, \vec{x}_s; x_s, \omega) \) and \( \hat{G}_s(\vec{x}_q, \vec{x}_p; x_s, \omega) \) are the decompositions of the Green’s function into beamlet domain. If we relate the \( \vec{\xi}_j \) and \( \vec{\xi}_p \) to the local incident and scattering angles using equation (6), \( E(\vec{\theta}_j, \vec{\theta}_p, x, z, \omega) \) is the efficacy of the acquisition system to the local scattering measurement. For the ideal case of infinite apertures, it should hold

\[
E(\vec{\theta}_j, \vec{\theta}_p, x, z, \omega) \equiv 1, \quad \text{for all } \vec{\theta}_j \text{ and } \vec{\theta}_p.
\] (23)

Therefore the value of \( E(\vec{\theta}_j, \vec{\theta}_p, x, z, \omega) \) is an indication on how the acquisition aperture and the overlaying structure influence the scattering measurements and imaging process in the target area. As examples, here we calculate the AAE matrices for the same acquisition system (201 shots with 176 left-hand receivers per shot) as for the LIM calculation in Figure 7 in a homogeneous background. The corresponding local AAE matrices are shown in Figure 8. Comparing with Figure 7, we can see the difference and relation of the local AAE matrices and the local image matrices (LIM). AAE matrices at different imaging points indicate the capability of the acquisition systems, while LIM’s are the results of the imaging process which utilizes that capability. The imaging result (local image matrix, e.g. Figure 7) can be considered as a convolution of the system response (AAE, e.g. Figure 8) with the scattering characteristics of the target (scattering matrix, e.g. Figure 6). Obviously, the larger the acquisition aperture, the more scattered energy can be received, resulting in longer lengths of the line segments of the local image matrices for the planar reflectors. We see also that deep targets are affected more seriously by the acquisition aperture than shallow targets.

For planar reflectors, we can reduce the AAE matrix to an acquisition dip response (ADR) vector:

\[
ADR(\vec{\theta}_n, x, z, \omega) = \sum_{\vec{\theta}} E(\vec{\theta}_n, \vec{\theta}_r, x, z, \omega).
\] (24)

The above procedure can be illustrated in Figure 9: all the contributions from various incident-receiving angle pairs for the same dip angle are summed together to get the total response for that dip angle. \( ADR(\vec{\theta}_n, x, z, \omega) \) measures the dip-angle response of the acquisition system, including the source and receiver apertures, at the target area.

We can get the ADR map for pre-selected source and receiver arrays or for the whole acquisition system. Averaging over a frequency band centering at \( \omega_0 \) can be done as

\[
ADR(x, z, \vec{\theta}_n, \omega_0 \pm \Delta \omega) = \left( \sum_{\omega_0 - \Delta \omega}^{\omega_0 + \Delta \omega} ADR^2(x, z, \vec{\theta}_n, \omega) \right)^{1/2}.
\] (25)

The ADR-map can be also calculated from the corresponding DI-maps.
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\[ ADR(x, z, \bar{\theta}_a; x_S, \omega_0 \pm \Delta \omega) = \sum_{S} \left[ D_a^2 \left( x, z, \bar{\theta}_a; x_S \right) \sum_{R} D_a^2 \left( x, z, \bar{\theta}_a; x_{R_S} \right) \right]^{1/2}, \]  

(26)

where \( \bar{\theta}_a, \bar{\theta}_r \) and \( \bar{\theta}_n \) are related through (19). The ADR-map in (26) is for a single source \( x_S \) with \( N_R \) receivers \( x_{R_S} \), which in fact is the receiver-aperture efficacy. For group sources or the whole acquisition aperture (\( N_S \) sources are considered), the ADR-maps can be obtained simply by summing up the ADR’s of individual sources

\[ ADR(x, z, \bar{\theta}_a) = \left[ \sum_{S=1}^{N_S} ADR^2 \left( x, z, \bar{\theta}_a; x_S \right) \right]^{1/2}. \]  

(27)

Analysis of acquisition-aperture efficacy as well as directional illumination for a given acquisition system can be achieved through investigating the DI and ADR distributions for the dominant frequency. This can increase the efficiency of the procedure dramatically. In the following, we will consider only the dominant frequency for both DI-map and ADR-map calculations.

Similar with the DI-map, the ADR-map can also be presented in the form of ADR gathers and ADR album with different dip angles. Figure 10 and Figure 11 show the ADR gathers and ADR album, respectively, obtained from the whole acquisition system for the lens model. In Figure 10, a rose diagram with 11 petals is plotted at each point. Different from those in the DI maps (Figure 2), each petal here represents the response of a local reflector with that dip direction. The length of the petal is proportional to the response strength. Note that the dip of a local reflector is defined as the angle with respect to the horizontal (clockwise), so the dip direction is perpendicular to the reflector normal \( n_\theta \). The total strength of the responses for all the directions is given by the number near each point. From Figure 10 we can see that with the same sources, the ADR-maps illustrate different directional features compared with the corresponding DI-maps (Figure 2). The local ADR’s represent the system responses to local reflectors, which also include the influences of the receiver apertures in addition to source illumination. The difference can be seen clearly in the case of horizontal reflectors. In Figure 11, the blind area for horizontal reflectors (\( \text{dip} = 0^\circ \)) under the lens has different shape from the shadow zone of vertical illumination map for the sub-lens region (Figure 3). Since the dip-response of a reflector is the sum of received reflected energies, horizontal reflectors are more sensitive to the influence of receiver aperture. However, The ADR albums for other dip-angles are not too different from the corresponding illumination maps. This indicates that normally incident illumination plays an important role for large dip-angle reflectors.

Application to The Seg-Eage Salt Model

We use the SEG-EAGE 2D salt model (Figure 12a) as another example for the application of directional illumination and acquisition efficacy analysis. The whole acquisition system of this model consists of 325 shots with 176 left-hand-side receivers for each shot, i.e. \( N_S = 325 \) and \( N_R = 176 \). Only the dominant frequency \( f_0 = 15\text{Hz} \) is considered for the calculations. The image by G-D beamlet prestack depth migration is plotted in Figure 12b (See Wu and Chen, 2001, 2002a).
In Figure 13, the directional illumination gathers of individual point sources and point source arrays are shown in the form of “rose diagram”, respectively. Obviously, these DI-maps of single or group sources illustrate the directivity features of source illumination and the influence of the salt body to subsalt illumination. The DI-map album for the whole acquisition aperture (total 325 shots) \( D_a(x,z,\theta, \omega) \) with \( a = 800 \text{m} \) is calculated for different local incident angles \( \theta \) and shown in Figure 14. We can see that although the intensity of the vertical illumination (Figure 14a) seems distributed more uniformly at the subsalt area, the oblique illuminations look highly non-uniform and shadow zones appear under the salt body, especially in the large angle cases (Figure 14c and 14e).

Next, we apply acquisition efficacy analysis and calculate the acquisition dip-response (ADR) for the model. In order to see the features of the AAE matrix for this model and the associated acquisition system, we select 4 deep target points of which P2, P3 and P4 are in the subsalt region, and P1, outside the subsalt as a reference point. We see that the local AAE matrix at P1 is rather normal (left panel in Figure 15). For the local AAE matrix, the \( x \)-axis is \( \theta_x (-\pi/2 - \pi/2) \), the \( z \)-axis is \( \theta_z (-\pi/2 - \pi/2) \) (see Figure 6). Since the receiver array is extended to the left-hand side of the corresponding source only, so the matrices are not fully filled. For the subsalt points P2, P3 and P4, the local AAE matrices are quite different from that of the reference point P1. For P2, the acquisition is more effective for left-hand-side incident and left-hand-side scattered waves; while for P4 the opposite is true, i.e. the AAE is better for right-side incident, right-side scattered waves. For P3, vertical or small-angle incident-scattered waves are favorable. From these local AAE matrices, we can further calculate and analyze the dip-responses for planar reflectors. The ADR album for the whole acquisition system is shown in Figure 16. At the subsalt area, the horizontal dip-angle response (Figure 16a) is relatively uniform compared with oblique dip-angle cases, which predicts that most horizontal structures can be well imaged by prestack migration. On the contrary, for the oblique structures, especially for the subsalt steep faults, the dip-angle responses have many “blind areas” due to the influences of the acquisition apertures and the overlaying salt body structure. The dip-angle responses for the steep subsalt faults are relative weak (Figure 16b and c) and the blind areas for steep reflectors with opposite signs of dip-angle are quite different. Blind areas for one dip may become “bright” areas for the other dip (compare Fig 16b, c with Figure 16d, e). Therefore, ADR mapping could be very useful for acquisition and migration designs.

From Figure 12b, we see that the three subsalt steep faults are poorly imaged. Many authors suggested that this is caused by poor subsalt illumination. However, if we consider only the total illumination and total acquisition efficacy, the puzzle still cannot be solved. From the total illumination and total dip-response of the acquisition system shown in Figure 14f and Figure 16f, respectively, we see that in the regions of the three steep faults, neither total illumination nor total dip-response is very weak. The poor image quality of the steep faults is hard to be explained by these figures. However, if we look at the directional illumination and dip-responses, the correlation between image quality and acquisition dip-response (ADR) becomes quite clear. Figure 17 gives the zoomed-in ADR’s for 0° (horizontal reflectors) and 45° (45°-dip reflectors) together with the subsalt image. We see excellent correspondence between dip-responses and the related reflector images. Note that the steep fault in the middle of the subsalt region has very weak dip-response, especially for the upper half. This explains the weak image of the fault and the total absence of the upper half. The left steep fault has even weaker
dip-response. However, its reflection coefficient is 2.5 times larger than the middle one, so its image is stronger than the steep fault in the middle.

After obtaining the DI and AAE (or ADR), we can perform various acquisition aperture corrections to improve the image quality and conduct local AVA analysis or local inversion for medium parameters. Directional analysis (including AAE mapping) provides the basis for illumination correction to obtain unbiased image amplitudes representing the scattering strengths of local heterogeneities. This is an important topic of current research on wave-theory based “true-amplitude”, “true-reflection” imaging. AAE matrices provide information on angle-dependent acquisition aperture and propagation effects, which may be critical for reaching the final goal of true-reflection imaging.

Conclusions

Directional illumination (DI) and acquisition-aperture efficacy (AAE) analyses based on beamlet wave field decomposition and propagation are proposed. The method has been applied to high- and low-velocity lens models and the SEG-EAGE 2D salt model to demonstrate the feasibilities and features of the approach. Beamlet decomposition provides localizations in both space and direction of the wave field, and can provide more flexible and accurate DI and AAE analyses compared with the traditional illumination analysis. For the salt model, the influences of acquisition geometry, combining with the propagation effects of the complex salt body on the image quality of prestack depth migration are studied through analyses of the DI-maps and the ADR (acquisition dip response) maps. Good correspondence between the calculated ADR using the proposed method and the quality of the related images for subsalt reflectors with different dips demonstrates the validity and application potential of the method.

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APPENDIX

WINDOWED FOURIER TRANSFORM (WFT) AND GABOR-DAUBECHIES FRAME (G-D FRAME)

Windowed Fourier Transform (WFT)

For a function \( f(x) \) defined in the time-domain, its windowed Fourier transform (WFT) can be defined as

\[
 f(\bar{x}, \bar{\xi}) = \int dx f(x) w^*(x - \bar{x}) e^{-i\bar{\xi}x},
\]

where \( \bar{x} \) and \( \bar{\xi} \) are the localized time and frequency respectively. \( w(x) \) is a square-integrable window function centered at \( x = 0 \), and \( {}^* \) denotes the complex conjugate. The localized time-frequency domain parameterized by \( (\bar{x}, \bar{\xi}) \) is the wavelet domain. Unlike the Fourier transform \( f(\xi) \), which represents the coefficient of a harmonic wave with frequency \( \xi \), in WFT \( f(\bar{x}, \bar{\xi}) \) is the coefficient of a wavelet with a frequency-lobe centered at \( \bar{\xi} \) (local frequency) and occupies a window centered at \( \bar{x} \). The inverse WFT (reconstruction) is given by

\[
 f(x) = \frac{1}{2\pi N^2} \int d\bar{\xi} \int d\bar{x} f(\bar{x}, \bar{\xi}) w(x - \bar{x}) e^{i\bar{\xi}x},
\]

where \( N \) is the \( L^2 \) norm of the window,

\[
 N^2 = \|w\|^2 = \int |w(x)|^2 \, dx.
\]

For a Gaussian window,

\[
 g(x) = \pi^{-\frac{1}{4}} \exp\left(-\frac{x^2}{2}\right),
\]

and \( N^2 = \|g\|^2 = 1 \).

Gabor-Daubechies Frame (G-D Frame)

Since the WFT reconstruction is very time-consuming, many studies have been performed to obtain sparsely sampled, but still accurate reconstruction schemes. For the Gaussian window function, the discrete WFT is closely related to the canonical coherent states in physics. Daubechies showed that for stable reconstruction oversampling \( \Delta_x \Delta_\xi < 2\pi \) must hold, where \( \Delta_x \) and \( \Delta_\xi \) are the time and frequency sampling intervals respectively. \( \Delta_x \Delta_\xi \) measures the size of windowed Fourier atoms (Latticed coherent states). The windowed Fourier frame atoms are

\[
 g_{mn}(x) = e^{im\Delta_\xi x} g(x - n\Delta_x)
\]

(A-5)
and the frame coefficients of a function \( f(x) \) can be calculated as

\[
\langle f, g_{mn} \rangle = \int dx f(x) e^{-im\Delta_x} g^*(x - n\Delta_x).
\]  \hfill (A-6)

If \( g_{mn}(x) \) constitute a frame, where \( g(x) \) is a window function and \( \Delta_x\Delta_\xi < 2\pi \), there exists two constants \( A > 0, B > 0 \), such that

\[
A \int dx |f(x)|^2 \leq \sum_m \sum_n |\langle f, g_{mn} \rangle|^2 \leq B \int dx |f(x)|^2.
\]  \hfill (A-7)

Function \( f(x) \) can be reconstructed from the so-obtained frame coefficients:

\[
f = \sum_m \sum_n \langle f, g_{mn} \rangle \tilde{g}_{mn},
\]  \hfill (A-8)

where \( \tilde{g}_{mn} \) is the dual frame vector,

\[
\tilde{g}_{mn}(x) = e^{im\Delta_x} \tilde{g}(x - n\Delta_x).
\]  \hfill (A-9)

It has been proved that the dual frame vector is also a windowed Fourier frame vector. That means that the dual frame vectors are time and frequency translations with a new window. In the case of the dual frame equal to the original frame, the frame is then called tight frame. There have been various methods and algorithms developed to construct dual frames, such as the conjugate gradient iterations or pseudo inverse method (Daubechies, 1992; Mallat, 1998, Qian and Chen, 1996).

Compared to the critical sampling case \( (\Delta_x\Delta_\xi = 2\pi) \), frame decomposition is overcomplete. The representation by its coefficients contains redundant information. The moderate redundancy results in a robust and stable reconstruction. From wave propagation point of view, the tight-frame or nearly tight-frame representation with moderate redundancy leads to good localizations in both time and frequency (or space and wavenumber), a very desirable feature for efficient extrapolation of wavefield. In our application, we decompose the frequency-domain wave field along the spatial axes using the G-D frame theory. In this case, the localized space-wavenumber domain parameterized by \( (\tilde{x}, \tilde{\xi}) \) is the beamlet domain. Unlike the Fourier transform \( f(\xi) \), which represents the coefficient of a global plane wave with wavenumber \( \xi \), in G-D frame decomposition, \( f(\tilde{x}, \tilde{\xi}) \) is the coefficient of a beamlet (small beam) with a lobe centered at \( \tilde{\xi} \) (local wavenumber) and occupies a window centered at \( \tilde{x} \). In wave propagation in beamlet domain, even when the dual frame is very close to the original frame in the nearly tight case, the error accumulation during wave propagation is still noticeable if the dual frame is replaced by the original frame. Therefore dual frame vector must be used for wave propagation and imaging. It is known than in oversampling cases, frame decomposition is redundant, and the dual frame is not unique. In our work, we select the dual frame window whose shape is the closest to the original frame window in the sense of the least square error and precompute it for each application using the method of Qian and Chen (Qian and Chen, 1996).

For simplicity, the above-described Windowed Fourier Frames with Gaussian window was called as Gabor-Daubechies (G-D) frame (Wu and Chen, 2001, 2002a). In Figure A1 we plot the Gaussian window and its dual frame windows for the cases of redundancy ratio equal to 2 and 4, respectively. The higher the redundant ratio, the closer the dual frame window function is to the original window function.
Figure Captions

Figure 1. Cartoons illustrating the concept of directional illumination maps. (a) For single-source case; (b) For multi-source case.

Figure 2. DI maps for all the 257 shots in the form of “Rose diagram” for the lens model ($f_0 = 15$Hz). Two velocity contrasts are considered: (a) $+100\%$ contrast; (b) $-50\%$ contrast.

Figure 3. DI (Directional Illumination) album for all the 257 shots for the lens model ($f_0 = 15$Hz). (a) for the high velocity lens ($+100\%$ velocity contrast); (b) for the low velocity lens ($-50\%$ velocity contrast)

Figure 4. Local scattering matrix and local image matrix.

Figure 5. Definition of local wavenumber vectors: $\vec{k}_i$ -- incident wavenumber vector, $\vec{k}_s = -\vec{k}_i$; $\vec{k}_g$ -- scattering wavenumber vector; $\vec{k}_e$ -- exchange wavenumber vector, $\vec{k}_e = \vec{k}_g - \vec{k}_i = \vec{k}_g + \vec{s}$. $\hat{s}$, $\hat{g}$, $\hat{n}$ are the corresponding unit vectors of $\vec{k}_s$, $\vec{k}_g$, $\vec{k}_e$, respectively.

Figure 6. (a) Geometry of a scattering experiment for a planar reflector; (b) Angle distribution of reflected energy for full aperture (-$\pi$ - +$\pi$) and half aperture (-$\pi/2$ - +$\pi/2$).

Figure 7. Local image matrices for different scatterers in a homogeneous medium (total 201 shots with 176 left-hand receivers): left panel: point scatterer; right panel: planar reflector.

Figure 8. Local acquisition aperture efficacy (AAE) matrix in a homogeneous medium (total 201 shots).

Figure 9. Illustration of acquisition dip-responses (ADR) calculated from local Acquisition aperture efficacy (AAE) matrix.

Figure 10. Acquisition dip-response (ADR) maps in the form of “rose diagram” for the lens model. Total 257 shots with 176 left-hand-side receivers for each shot are used. (a) high-velocity lens; (b) low-velocity lens.

Figure 11. Acquisition-dip-response (ADR) album of the lens model from all the 257 shots with 176 left-hand-side receivers for each shot: (a) for high-velocity lens ($+100\%$ velocity perturbation); (b) for low-velocity lens ($-50\%$ velocity perturbation).

Figure 12. (a) SEG-EAGE 2D salt model; (b) the image by G-D beamlet prestack depth migration.
Figure 13. Directional illumination (DI) maps for the dominant frequency (15Hz) in the form of “rose diagram” for the SEG-EAGE salt model: (a) from a single shot at (271,0); (b) from a single shot at (451,0); (c) from a shot array of 40 shots at every other points from (201,0) to (279,0); (d) from a shot array of 40 shots at every other points from (431,0) to (509,0).

Figure 14. Directional illumination (DI) album for the dominant frequency (15 Hz) from all the 325 shots for the SEG-EAGE salt model: (a) vertical direction; (b) -30° (from vertical to left); (c) -45° (from vertical to left); (d) +30° (from vertical to right); (e) +45° (from vertical to right); (f) total illumination intensity.

Figure 15. Local acquisition efficiency (AAE) matrices for point scatterers in the SEG-EAGE salt model (total 325 shots with left-hand 176 receivers per shot).

Figure 16. Acquisition-dip-response (ADR) album of the SEG-EAGE salt model from all the 325 shots with 176 left-hand-side receivers for each shot: (a) horizontal dip; (b) +30° (down from horizontal); (c) +45° (down from horizontal); (d) -30° (up from horizontal); (e) -45° (up from horizontal); (f) total response intensity.

Figure 17. Image of G-D beamlet prestack depth migration (top), in comparison with the acquisition dip-response (ADR) for horizontal (middle) and 45°-dip (bottom) reflectors. We see good correlation between the ADR’s and the reflector images of the corresponding dips.

Figure A-1. Gaussian window (dotted lines) and its dual frame windows (solid lines) for: (a) redundancy ratio = 2; (b) redundancy ratio = 4.
Directional Illumination Analysis

Figure 1:
Figure 2:

(a) +100% velocity contrast

(b) -50% velocity contrast
Figure 3:
Figure 4

Local Scattering Matrix (only dependent on the scatterer) $S(\theta_i, \theta_g)$

Local Image Matrix (includes aperture and propagation effects) $L(\theta_i, \theta_g)$

Figure 5:

$\vec{k}_S = -\vec{k}_i$

(scattering wavenumber vector)

$\vec{k}_N = \vec{k}_g - \vec{k}_i = \vec{k}_g + \vec{k}_S$

(exchange wavenumber vector)
Figure 6:

Figure 7:
Figure 8:

Figure 9:
Figure 10:

(a) +100% velocity contrast

(b) -50% velocity contrast
Figure 11
Figure 12
Figure 13:
Figure 14:
Figure 15:
Figure 16:
Figure 17:

G-D beamlet prestack migration

Acquisition dip-response (horizontal)

Acquisition dip-response (45° dip)
Figure A-1:
Wave-equation based seismic illumination analysis

Xiao-Bi Xie*, Shengwen Jin*, and Ru-Shan Wu*

ABSTRACT

We present a wave-equation based method for seismic illumination analysis. A one-way wave-equation based generalized screen propagator is used to extrapolate the wavefields from sources and receivers to the subsurface target. A local plane-wave analysis is used at the target to calculate localized directional energy fluxes for both source and receiver wavefields. We construct an illumination matrix using these energy fluxes to quantify the target illumination conditions. The target geometry information is used to manipulate the illumination matrix and generate different types of illumination measures. The wave-equation based approach can properly handle forward multiple-scattering phenomena including focusing/defocusing, diffraction and interference effects. It can be directly applied to complex velocity models. Velocity model smoothing and Fresnel-zone smoothing are not required. Different illumination measurements derived from this method can be applied to target oriented or volumetric illumination analyses. This new method is flexible and practical for illumination analysis in complex 2D and 3D velocity models with nontrivial acquisition and target geometries.

INTRODUCTION

The illumination of a subsurface target is affected by many factors, e.g., the limited acquisition geometry, the complex overburden structure and the reflector dip angle. An uneven illumination causes a distorted image. Seismic illumination analysis quantifies such image distortion and has many applications in seismic migration/imaging. The effect of acquisition geometry can be evaluated by calculating illuminations of different shooting patterns. More accurate amplitude variation with angle (AVA) or amplitude variation with offset (AVO) may be obtained if the observation is corrected with angle-dependent illumination.

In the past, the illumination estimate was based simply on the acquisition geometry at the surface under the assumption of a homogeneous velocity model, horizontal targets and symmetric raypaths (Hoffmann, 2001). These assumptions may be invalid for complex structures under realistic situations. To properly calculate the target illumination, we have to extrapolate the
wavefield between sources, targets and receivers. In order to calculate the angle-dependent illumination, we also need directional information from the wavefield.

Traditionally, illumination and resolution analyses have used the ray-based method (Schneider and Winbow, 1999; Bear et al., 2000). The ray-based method can provide both intensity and directional information carried in the wavefield. Dynamic ray tracing is used to calculate energy propagation along the source-target-receiver path using the smoothed velocity model. The common reflection point (CRP) gathers (ray amplitude, hit count and offset coverage, etc.) on the target are used for the illumination measurements. A Fresnel-zone smoothing is usually applied to obtain smoothly distributed coverage on the target horizon (Muerdter and Ratcliff, 2001ab). These procedures have been discussed by Muerdter and Ratcliff (2001ab) and Muerdter et al. (2001), who made a comprehensive demonstration of the application of ray-based illumination analysis in the subsalt region. Using common focusing point (CFP) analysis, Berkhout et al. (2001) and Volker et al. (2001) investigated the effect of acquisition geometry on target illumination and migration resolution. Hoffmann (2001) used the illumination information for resolution analysis. Based on the illumination analysis in the local angle domain, Gelius et al. (2002) and Lecomte et al. (2003) defined a resolution function and discussed the effect of a complex velocity model on the illumination and resolution.

Although the ray-based illumination analysis can handle both irregular acquisition geometry and laterally varying velocity models, the high-frequency asymptotic approximation and the caustics inherent in ray theory may severely limit its accuracy in complex regions (Hoffmann, 2001). While the ray-based method is relatively efficient for target-oriented analysis, it is still not a cost-effective approach for full-volume 3D illumination analysis. Attempts have been made to apply the wave-equation based method to seismic illumination and resolution analysis. Schuster and Hu (2000) derived an analytical solution for target point scatter responses by assuming a homogeneous velocity model with continuously distributed sources and receivers. Rickett (2003) developed a normalization scheme to compensate for the effect of irregular illumination. However, these results were usually restricted to simple geometries or did not provide directional information that is crucial for target-oriented illumination.

Recently developed dual-domain one-way wave-equation based methods (e.g., Stoffa et al., 1990; Ristow and Ruhl, 1994; Xie and Wu, 1998, 2005; Jin et al., 1998, 2002; Huang et al., 1999; Xie et al., 2000; Wu and Chen, 2001; and Biondi, 2002) provide propagators for seismic wave extrapolation in complex velocity models. Although these propagators neglect reverberations between heterogeneous layers, they properly handle forward multiple-scattering phenomena including focusing/defocusing, diffraction and wave-interference effects. These algorithms alternate between the space domain and wavenumber domain using the Fast Fourier Transform (FFT). These methods make optimal operations in each domain, resulting in a fast and accurate extrapolation of the wavefield. This makes them suitable for seismic forward modeling and migration/imaging. However, the inability to provide localized angle information prevents them from being used for directional illumination calculations.

There have been attempts to calculate angle information using the wave-equation based methods. The offset plane-wave and related offset angle generated from offset domain migration have been used in the velocity updating (Prucha et al. 1999; Mosher et al., 2001). Such angle information does not directly relate to the wave propagation direction and cannot be used in
illumination analysis. To extract angle information from the wavefield, Xie and Wu (2002) proposed an approach based on a local plane-wave analysis. Through local slant stacking or a windowed Fourier transform, the approach provides localized angle information. Xie et al. (2003) tested the local plane-wave analysis method in illumination analysis. Wu and Chen (2002, 2003) and Wu et al. (2003) used wavelet transform theory (Gabor-Daubechie frame) to decompose the wavefield into components with localized angles for illumination analysis. Jin and Walraven (2003) applied the directional illumination to investigate the causes of subsalt imaging shadows.

In this research, we present an illumination analysis method using the generalized screen propagator (Xie and Wu, 1998) and the local plane-wave analysis (Xie and Wu, 2002; Xie et al., 2003). The wave-equation based propagator extrapolates the wavefields from sources and receivers to the target region. Local plane-wave analysis is conducted at the target position to obtain localized directional energy fluxes for both source and receiver wavefields. From these energy fluxes, we construct a local illumination matrix to describe the target illumination. By manipulating the illumination matrix, different illumination measures can be calculated. We use several 2D and 3D numerical examples to demonstrate the potential applications of these illumination measurements.

METHODS

Consider using a survey system composed of a source located at \( r_s \) and a receiver located at \( r_g \) to investigate a small subsurface target region \( V(r) \) in the vicinity of location \( r \) (see Figure 1). The source sends a seismic wave to the target. Within the target region, the incident wave interacts with the reflector and generates a reflected or scattered wave that propagates from the target to the receiver. Using the multiple-forward scattering/single-backscattering approximation, the seismic wave at the receiver can be expressed as

\[
u(r_s, r_g) = 2k_0^2 \int_{V} m(r') G(r'; r_s) G(r'; r_g) dr',
\]

where \( r' \) is a local coordinate within \( V(r) \), \( m(r') = \delta r / c(r') \) is the velocity perturbation, \( c(r) \) is the velocity, \( k_0 = \omega / c_0(r) \) is the background wavenumber, \( c_0(r) \) is the local background velocity inside \( V(r) \), \( \omega \) is the angular frequency, and \( G(r'; r_s) \) and \( G(r'; r_g) \) are Green’s functions with sources at \( r_s \) and \( r_g \), respectively. The reciprocity theorem \( G(r'; r_s) = G(r_s; r') \) has been used. For simplicity, the apparent frequency dependence has been omitted from equation 1 and all the following equations. Considering that \( V \) is small and the Green’s functions mostly propagate in the background velocity medium, we choose to use one-way wave-equation based propagators (Xie and Wu, 1998) to calculate these Green’s functions.

Applying the local plane-wave decomposition (Xie and Wu, 2002; Xie et al., 2003) within \( V \), the Green’s functions are decomposed as

\[
\begin{align*}
G(r'; r_s) &= \int G(K, r'; r_s) e^{ik' r} dK, \\
G(r'; r_g) &= \int G(K, r'; r_g) e^{ik' r} dK.
\end{align*}
\]
By substituting equation 2 into equation 1, we have

\[ u(r, r_s, r_g) = 2k_0^2 \int \int G(K_s, r; r_s) G(K_g, r; r_g) m(r, k_g + k_s) dK_g dK_s, \]  

(3)

where

\[ m(r, k_g + k_s) = \int m(r') e^{i(k_g + k_s) r'} dv'. \]  

(4)

In equations 2 to 4, \( k = K + k_z \hat{z} \) is the local wavenumber, \( K \) is the horizontal wavenumber, \( k_z \) is the vertical wavenumber, \( \hat{z} \) is the vertical unit vector, \( k_s = K_s + k_z \hat{z} \) and \( k_g = K_g + k_z \hat{z} \) are local transforms with respect to \( r' \) (not \( r_s \) and \( r_g \)), and \( k_{z, s} \) and \( k_{z, g} \) are vertical components of \( k_s \) and \( k_g \), respectively. Subscripts \( s \) and \( g \) denote the source-side and receiver-side wavefields, respectively. Given the local reference velocity \( c_0(r) \), the components of the wavenumber are not fully independent, i.e., \( k_s^2 + k_g^2 + k_z^2 = k_0^2 \), and \( k_z \) can be determined from its horizontal component via \( k_z = \sqrt{k_0^2 - K^2} \). We retain the parameter \( r \) in these equations to indicate the location of the target region. Equation 3 links the observation with the source and subsurface target and forms the basis of many seismic methods (e.g., seismic modeling, migration, inversion and tomography). Taking equation 3 as the starting point, we adopt the mean square of the Green’s function to define the illumination. Usually, a fictitious plane reflector with a dipping angle and a unit (or constant) reflectivity is required for calculating the illumination (Muerdter and Ratcliffe, 2001a). To generalize this into an arbitrarily non-flat reflector, we substitute \( m(r, k_g + k_s) \), which can be regarded as a wavenumber-domain local reflectivity, with its normalized amplitude spectrum. A target illumination response function is then defined as

\[ D(r, r_s, r_g) = \int \int A(r, K_s, K_g; r_s, r_g) I(r, k_g + k_s) dK_g dK_s, \]  

(5)

where \( M(r, k) = |m(r, k)| \), and

\[ A(r, K_s, K_g; r_s, r_g) = 2k_0^2 I(K_s, r_s; r) I(K_g, r_g; r) \]  

(6)

is the local illumination matrix of the source-receiver pair \( (r_s, r_g) \). The equations

\[ I(K_s, r; r_s) = G(K_s, r; r_s) G^*(K_s, r; r_s) \]  

(7)

and

\[ I(K_g, r; r_g) = G(K_g, r; r_g) G^*(K_g, r; r_g) \]  

(8)

are mean squares of the Green’s functions, which are proportional to energy fluxes from the source and receiver to the target, respectively. The superscript “*” denotes complex conjugation.

For a system composed of multiple sources and receivers, the illumination response can be calculated by stacking contributions from individual source-receiver pairs. From equation 5,
\[D(r) = \sum_{r_s} \sum_{r_g} D(r, r_s, r_g) = \iint A(r, K_s, K_g) M(r, k_g + k_s) dK_s dK_g,\]  

(9)

where

\[A(r, K_s, K_g) = \sum_{r_s} \sum_{r_g} A(r, K_s, K_g; r_s, r_g)\]  

(10)

is the local illumination matrix for the entire acquisition system. The summations over \(r_s\) and \(r_g\) are based on the acquisition geometry. In equation 9, for a given acquisition geometry and background velocity model, the matrix \(A(r, K_s, K_g)\) is composed of all possible local scattering events \((k_s, k_g)\) that may contribute to the target illumination. For a particular local target structure, \(M(r, k)\) provides the mapping relationship between the incident and scattered waves and manipulates energy within the local illumination matrix. The integral sums the energy that can actually contribute to the particular target and gives the illumination response at location \(r\). The effects of acquisition configuration, the background velocity model and target geometry are all included in the calculation. The illumination response function is calculated using the mean square of the amplitude. When the amplitude is preferred, the root mean square (RMS) of \(D(r)\) can be used.

For a 3D velocity model, equations 9 and 10 are defined in the acquisition wavenumber space \((k_s, k_g)\), resulting in a 4D illumination matrix. The illumination matrix can be defined in the target spectrum space through the transformations (see Figure 1)

\[k_d = k_g + k_s,\]

\[k_r = k_g - k_s,\]  

(11)

or

\[K_d = K_g + K_s,\]

\[K_r = K_g - K_s.\]  

(12)

Note that after the transformations, the lengths of vectors \(k_d\) and \(k_r\) are no longer of constant value \(k_0\). To determine their vertical components, we have to go back to \(k_s\) and \(k_g\). The illumination matrix can also be defined using two sets of directions (angles) \((\theta_s, \phi_s)\) and \((\theta_g, \phi_g)\) are incidence and scattering directions, and \(\theta\) and \(\phi\) are the corresponding dip and azimuth angles. The relationship between the wavenumber and angles is

\[k = k_0 (\sin \theta \cos \phi \hat{e}_x + \sin \theta \sin \phi \hat{e}_y + \cos \theta \hat{e}_z).\]  

Therefore, equation 9 can be expressed using either \((K_s, K_g)\), \((K_d, K_r)\) or \((\theta_s, \theta_g)\) with proper Jacobians included.
For a locally planar dipping structure (i.e., the Kirchhoff scattering model) given by
\[ m(r') \sim \delta(r' - n) \], we have \[ M(k) \sim \delta(k - Cn) \] and \[ k_g + k_s = Cn \], where \( n \) is the normal vector of the reflector, and \( \delta \) is Dirac’s delta function. Since \( |k_s| = |k_g| = k_0 \), we know \( n \) is the angle bisector of \( k_g \) and \( k_s \), (see Figure 1), and \( C = 2k_g \cdot n = 2k_s \cdot n = 2k_0 \cos i \) with \( i \) as the reflection angle. Substituting this \( M(k) \) into equation 9 yields

\[ D(r, n) = \int \int A(r, K_s, K_g) \delta(K_g + K_s - CN) dK_s dK_g = \int A(r, CN - K_g, K_g) dK_g. \] (13)

where \( N \) is the horizontal component of normal vector \( n \). Since \( n \) is a unit vector, it is determined by \( N \). \( D(r, n) \) is the acquisition dip response (ADR) which gives the illumination response for a plane reflector located at \( r \) with a dipping \( n \). Equation 13 implies that a plane reflector is exclusively illuminated by mirror reflections. Using transform 11 we see that \( k_d = Cn \) is linked to the dipping direction and that \( |k_s| = 2k_0 \sin i \) is linked to the reflection angle. Considering an isotropic point scatter (i.e., the Born scattering model) \( m(r') \sim \delta(r') \), we have \( M(k) \sim \text{const} \). Substituting \( M(k) \) into equation 9 yields

\[ D_T(r) = \int \int A(r, K_s, K_g) dK_s dK_g. \] (14)

\( D_T(r) \) is the total illumination response which sums energy from all possible scattering combinations.

The illumination responses for other structures, such as a reflector with a particular curvature or roughness or a layer with random velocity fluctuations, can be obtained using a different \( M(k) \). Due to the small size of region \( V \), a simple geometry is usually sufficient for illumination analysis.

**NUMERICAL EXAMPLES OF VARIOUS ILLUMINATION MEASUREMENTS**

In numerical implementations, the formulations can be discretized. The size of the target region \( V \) for the decomposition should be small enough to maintain localized properties of both the wavefield and the model, while large enough to preserve the internal structures of the wavefield and the model (e.g., the wave propagation direction and the dip of the structure). The local plane-wave decomposition of equation 2 can be conducted on horizontal coordinates by using either local slant stacking or a windowed FFT (Xie and Wu, 2002, Xie et al., 2003). This geometry is consistent with most of the one-way wave-equation based propagators that choose the vertical direction as the primary propagation direction and apply the transform in the horizontal direction. With these one-way propagators, we can use the depth step as the vertical dimension of \( V \) and use the wavelength to determine the horizontal size. The equations developed in this section are in the frequency domain. To calculate illumination responses from multiple frequencies, the response from each individual frequency should be weighted by the source spectrum and summed together. To demonstrate the application of the wave-equation
based illumination analysis, various types of illumination measurements for both 2D and 3D are calculated.

**Local illumination matrix**

The illumination matrix can be calculated using equation 10. For a 2D model, the horizontal wavenumber is scalar, and the local illumination matrix \( A(r, K_s, K_g) \) becomes a 2D matrix. Figure 2 is a sketch showing the structure of a 2D local illumination matrix. The horizontal and vertical axes denote the horizontal components \( K_s \) and \( K_g \) of incidence and scattering wavenumbers, or equivalently, incidence and scattering angles \( \theta_s = \sin^{-1}(K_s/k_0) \) and \( \theta_g = \sin^{-1}(K_g/k_0) \). The main and auxiliary diagonal directions are horizontal reflection wavenumber \( K_r \) and dipping wavenumber \( K_d \), respectively. As mentioned above, each element \( (K_s, K_g) \) in the matrix corresponds to an independent scattering observation of the target. A strip parallel to the vertical direction consists of scatterings with a common incidence angle. Conversely, a strip along the horizontal direction corresponds to scatterings with a common scattering angle. Energy distributed within a strip parallel to the main diagonal is composed of all mirror reflections that contribute to the illumination of a plane dipping reflector. Similarly, energy distributed within a strip parallel to the auxiliary diagonal corresponds to scatterings that have a common reflection angle. The coordinate transformation 12 rotates the 2D scattering matrix 45 degrees. The energy distribution in the illumination matrix gives the effective acquisition aperture at a local target. To recover the entire target spectrum, properly distributed energy in the illumination matrix is preferred.

As an example, Figure 3 gives local illumination matrices at selected locations in the 2D SEG/EAGE salt model (Aminzadeh et al., 1997). The normalized value \((A/A_{\text{max}})^{1/2}\) is used in the figure, with \( A_{\text{max}} \) as the maximum value in the model. The energy occupies the upper-left corner within the matrix due to the off-end data acquisition for this model. At a shallow depth, the model is illuminated by wider effective apertures, but the energy spans a relatively narrow aperture at deeper depths. Due to the shadowing effect within the subsalt region, the energy is generally weak and apparently missing for certain dipping and reflection angles. This is the main cause of poor imaging in the subsalt region.

**Wavenumber domain illumination**

As mentioned above, the illumination matrix can be expressed in the target wavenumber coordinate \( (k_d, k_r) \) (see Figures 1 and 2). By integrating \( A(r, k_d, k_r) \) over \( k_r \), we project the illumination matrix onto the target spectrum space,

\[
A_s(r, k_d) = \int A(r, k_d, k_r) \, dk_r, \quad (15)
\]

where \( A(r, k_d, k_r) \) is obtained from \( A(r, K_s, K_g) \) through transformation 12, and the latter can be derived from \( A(r, K_s, K_g) \) using
\[ A(r, k_s, k_g) = A(r, K_s, K_g) \delta\left( k_{sz} - \sqrt{k_{sz}^2 - K_{sz}^2} \right) \delta\left( k_{gz} - \sqrt{k_{gz}^2 - K_{gz}^2} \right), \]  

(16)

where \( k_{sz} \) and \( k_{gz} \) are vertical components of \( k_s \) and \( k_g \). In the processes of seismic imaging and diffraction tomography, properly distributed illumination in the target spectrum space is essential in order to obtain high resolution results. Several authors (e.g., Wu and Toksöz, 1987; Woodward, 1992) have indicated the importance of wavenumber domain illumination. For a given acquisition system and velocity model, illumination in the target spectrum domain can be calculated using equation 15. Similarly, by integrating \( A(r, k_s, k_g) \) over \( k_s \), the illumination matrix is projected to the reflection wavenumber domain,

\[ A_r(r, k_s) = \int A(r, k_s, k_g) dk_s, \]  

(17)

which gives the reflection angle coverage for all dipping angles at the target location. Figure 4 shows the wavenumber domain illumination matrices \( A_d(r, k_s) \) and \( A_r(r, k_s) \), respectively.

Shown in Figure 5 is the illumination matrix \( A_r(r, k_s) \) at selected locations in the 2D SEG/EAGE salt model. In sediments at the shallow part, good illumination extends to a wide dipping angle range. The dipping angle coverage deteriorates with increasing depth, particularly in the subsalt region. In the subsalt region, the acquisition system provides limited vertical resolution and poor horizontal resolution, which explains the poor image quality of steep faults.

**Illumination as a function of target reflection angle**

For a target reflector with a local normal vector \( n = n(r) \), substituting \( K_s = CN(r) \) into the illumination matrix \( A(r, K_s, K_g) \), we derive the target illumination \( A[r, CN(r), K_g] \) as a function of dipping \( n \) and reflection angle \( i = \sin^{-1}(K_g/2k_0) \). Figure 6 shows the reflection angle distribution of the illumination along four target horizons in a 2D constant velocity model. One hundred and eighty-one surface shots are used to generate the illumination. One of these shots and its 3000 m long cable are indicated in the figure. The four reflectors (Baina et al., 2002) are labeled from \( T_1 \) to \( T_4 \), in which \( T_1 \) has the largest variation in dipping angles with a maximum value of approximately 40 degrees, \( T_2 \) has a constant dipping angle, \( T_3 \) has a moderate change in dipping angles and \( T_4 \) is a horizontal reflector. The fan-shaped patterns on these reflectors represent the normalized illumination intensities as a function of reflection angle. Note that the angle coverage decreases with increasing depth. The sections with steep dips have reduced angle coverage and weak intensities. We deliberately choose a constant background velocity model to show that the illumination is affected by both acquisition aperture and target dipping even in simple scenarios. To obtain correct amplitude variation with angle (AVA), compensation of the illumination versus reflection angle is crucial and should be taken into account.

**Target-oriented illumination**

The acquisition dip response (ADR) \( D[r, n(r)] \) on a target can be calculated using equation 13 by substituting the target normal vector \( n = n(r) \) into the integration. Figure 7 compares the amplitudes of prestack depth images and computed illuminations on targets for both constant and variable velocity models. The variable velocity model and the target reflectors
are given by Baina et al. (2002) with all reflectors having a constant reflectivity. The constant velocity model is created by assigning a velocity of 3.5 km/sec throughout the model. In Figure 7, the left column is for the constant velocity model with (a) the velocity model and target reflectors, (b) prestack depth image, (c) amplitudes picked from the depth image, (d) target ADRs, and (e) the target illumination coverage as a function of reflection angle. The horizontal coordinates are horizontal distance and have been aligned with the same scale. The results reveal that, even for a constant velocity model, the image amplitude varies dramatically due to the joint effect of limited acquisition aperture and dipping of the target. The minimum image amplitude corresponds to sections with the steepest dips, where the illumination result shows the narrower reflection angle coverage. The target ADRs predict the variations of the image amplitudes very well. Such information can be further used as the basis for checking the image quality and correcting the image amplitude. The right column is similar to the left column except that the varying velocity model is used. In this case, the image amplitude becomes more complex. The illumination analysis, however, still properly predicts the image amplitude.

**Volumetric illumination analysis**

The volume ADR map, which can be calculated using equation 13, represents the acquisition dip response as a function of space with the dipping angle as a parameter. Shown in Figure 8 is the $30^\circ$ ($n \cdot \hat{e}_z = \cos 30^\circ$) ADR map for the 3D SEG/EAGE salt model (Aminzadeh et al., 1997). The normalized value $\left[ D(r, n)/D_{\text{max}} \right]^{1/2}$ is used in the figure where $D_{\text{max}}$ is the maximum value. In this model, the acquisition system is composed of a source and four cables of 3200 m length. Figure 8a is the ADR in a vertical profile including the source and cables, while Figure 8b shows the ADR in a horizontal slice at a depth of 1000 m. These ADR maps reveal the relationship between the acquisition system, the velocity model and the illumination. Figure 9a shows an in-line section of the 3D SEG/EAGE salt velocity model. Figure 9c shows the normalized vertical ($n = \hat{e}_z$) ADR of the same section. The corresponding depth image is shown in Figure 9b, which is obtained by a 3D prestack offset-domain wave-equation migration with local reference velocities (Jin et al., 2002). Image shadows are present beneath the salt body, including the missing events on the horizontal base line. The vertical ADR associated with the illumination of horizontal events shows poor illumination in the same target area. Figure 10 shows a similar relationship between the illumination and the depth image for a horizontal slice at 2500 m depth. Strong total ADR illumination corresponds to the high quality image at the left side of the model, while a poor image is located in the weak illumination zone.

**Summary of different illumination measurements**

Table 1 shows various illumination measurements with different levels of detail. Towards the top of the table are high-order illumination matrices which contain more information but are composed of larger data sizes. More illumination measurements can be derived from these matrices with additional constraints. Towards the bottom of the table, lower-order measurements consist of more condensed information with compact data sizes. Different levels of measurements can be chosen to meet the specific purposes of the illumination analysis. The measurements with extensive information can be applied to selected locations for comprehensive investigations, while measurements with condensed information can be used to investigate spatial variations of illuminations, such as target-oriented or volumetric analysis.
DISCUSSION

In previous work on illumination analysis (e.g., Wu and Chen, 2002, 2003; Xie and Wu, 2002; Xie et al, 2003), the illumination is formulated for plane reflectors using source and receiver beams with mirror reflections. In this paper, the information regarding the target spatial spectrum is introduced in the formulation. Various illumination measurements can be obtained within this framework. The illumination results can be linked to other analyses, such as diffraction tomography and resolution analysis.

At present, the full-wave method is still too expensive for illumination analysis. The one-way propagator is used instead to calculate the wavefield from sources and receivers to the target. Some approximations are introduced into the one-way propagators. Transmission losses are often neglected. Luo et al. (2004) discussed the difference between illuminations calculated using one-way and full-wave methods and noted that the major factors affecting the illumination are limited acquisition aperture, dipping of the reflector and accuracy of the propagator. Among these three factors, the first two are usually more crucial than the accuracy of the propagator.

In contrast to seismic imaging, illumination analysis only uses the amplitude of the wavefield. Except for the plane-wave decomposition, the phase information is not used. Therefore, reducing the number of sources and receivers does not cause spatial aliasing, and will not seriously affect the illumination calculation. Unlike two-point ray-tracing, the wave-equation based propagator can simultaneously extrapolate waves into a large model space. With a frequency-domain propagator, we can calculate illumination for a number of frequencies or for a single dominant frequency. The result usually provides a satisfactory illumination estimate. The above mentioned approximations can reduce the CPU time considerably and make the wave-equation based approach an efficient tool for volumetric illumination analysis.

The equations formulated in this paper are based on one source-receiver pair. Complex acquisition geometries (locations of sources, number of streamer cables, receiver locations, navigation directions, etc.) can be composed by summing contributions from multiple source-receiver pairs. Since individual source and receiver wavefields are calculated independently, the wave-equation method is capable of computing illuminations from an irregularly distributed acquisition system.

CONCLUSIONS

A wave-equation based method was developed for seismic illumination analysis. Various illumination measurements derived from this method can be used to optimize the acquisition survey design, evaluate the image quality, and make corrections to the seismic image, resulting in more accurate subsurface physical parameter retrieval. The current method has the following features.

The wave-equation based propagator is adopted to calculate the wave propagation. Angle-related information is extracted from the wavefield by local plane-wave analysis. Therefore, this method can properly extrapolate the wavefield in complex media. Velocity
smoothing and Fresnel-zone smoothing are no longer required as in the high frequency asymptotic ray-based approach.

The illumination analysis part of the method is independent of its wavefield extrapolation part. Therefore, we conclude that the illumination analysis can be applied to most one-way wave propagation methods (e.g., the generalized screen propagator and the one-way implicit finite-difference propagator), full-wave-equation methods (e.g., the full-wave finite-difference scheme), and ray-based or ray-beam-based methods (e.g., the Kirchhoff and Gaussian beam approaches). This gives us the flexibility to choose different propagators based on the trade-off between their efficiency and accuracy.

Illumination measurements with different levels of detail can be derived from this method. Based on the goal of investigation, we can conduct illumination analysis by choosing the optimal category of measurement. Our wave-equation based illumination analysis is more efficient than the ray-based method when calculating the volumetric illuminations. Therefore, we conclude that the wave-equation based illumination analysis provides a flexible and efficient tool to calculate target-oriented or volumetric illuminations in 2D and 3D complex models.

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REFERENCES


Table 1. Comparison of different illumination measurements.

<table>
<thead>
<tr>
<th></th>
<th>Illumination matrix</th>
<th>$A(r, K_x, K_y)$</th>
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<tr>
<td>Wavenumber domain illumination matrices</td>
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<tr>
<td>ADR</td>
<td>$D(r, n)$</td>
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<tr>
<td>Total illumination</td>
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<tr>
<td>Target reflection angle distribution</td>
<td>$A[r, n(r), K_r]_{r \in \text{target}}$</td>
<td></td>
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<tr>
<td>Target ADR</td>
<td>$D[r, n(r)]_{r \in \text{target}}$</td>
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Towards more extensive information and larger data size.
Figure 1. Sketch showing the coordinate system.
Figure 2. Sketch showing the structure of a 2D local illumination matrix. The horizontal and vertical coordinates are $K_r$ and $K_g$, respectively. The main and auxiliary diagonal directions are horizontal reflection wavenumber $K_r$ and dipping wavenumber $K_d$, respectively. Different angle gathers are shown as strips with different orientations.
Figure 3. Local illumination matrices at selected locations in the 2D SEG/EAGE salt model. In the sediments at shallow part, the model is illuminated by wider effective apertures. The illumination spans a relatively narrow aperture at deeper part, especially in the subsalt region.
Figure 4. Wavenumber domain illumination matrices in $k_x$ domain (left panel) and $k_z$ domain (right panel) for a 2D model.
Figure 5. Wavenumber domain illumination matrices $A_d(r, k_d)$ at selected locations in the 2D SEG/EAGE salt model. In the sediments at the shallow part, good illumination extends to a wide dipping angle range. In the subsalt region, the dipping angle coverage of the illumination is significantly reduced.
Figure 6. Illumination distribution as a function of reflection angle on the targets. The illumination angle coverage decreases with the increasing depth. The sections with steep dips have reduced angle coverage and their illumination intensities become weak.
Figure 7. Comparison between the images and illuminations on the targets. The left column is for a constant velocity model and the right column is for a variable velocity model. From top to bottom, different rows are (a) velocity models, (b) prestack depth images, (c) amplitudes picked from prestack depth image, (d) ADRs calculated from the illumination analysis, and (e) illuminations as functions of reflection angles. (f) to (j) are similar results for the variable velocity model. The horizontal coordinates are distance and have been aligned using the same scale.

Figure 8. The 30° ADR from a single shot in a 3D model. Shown in (a) is the illumination of a vertical profile, and shown in (b) is the illumination of a depth slice at 1000 m. The acquisition system consists of a source and four cables with 3200 m length as indicted in the figure.
Figure 9. Illumination analysis for an in-line section of the 3D SEG/EAGE salt model. (a) shows the velocity model, (b) is the depth image by 3D wave-equation prestack depth migration, and (c) shows the corresponding vertical ADR associated with the near-horizontal events. A poor illumination zone is present beneath the salt body. This is consistent with the image shadows where the subsalt structures, including the horizontal base line, are not well imaged.
Figure 10. Illumination analysis for a depth slice of the 3D SEG/EAGE salt model. (a) shows the velocity model at depth 2500 m, (b) is a depth image by 3D wave-equation prestack depth migration, and (c) shows the total ADR associated with the contributions from all possible dipping events. The depth image quality is superior at the left portion of the model which corresponds to the strong illumination, while the image is poor at the right portion which is related to the weak illumination.
Inverting the power spectrum for a heterogeneous medium

Yingcai Zheng and Ru-Shan Wu

Abstract

The delta-correlated assumption of the random medium in the vertical direction is of considerable significance in statistical wave propagation problems. Assuming the Rytov and parabolic approximations, this work is the first to exactly derive the Transverse Coherence Functions (TCFs) without using the delta-correlated approximation. The error introduced by this approximation is shown to be negligible when the source-receiver distance is large. Our new formulation shows that the TCF does have vertical resolution, in contrast to previous assertions. We define a new observational variable; its Fourier transform directly yields the Power Spectral Density Function (PSDF) of the random medium. This enables determination of the scale and statistical distribution of the heterogeneities without assuming the power spectrum function \textit{a priori}.

1. Introduction

The delta-correlated assumption of the random medium in the vertical direction has been used extensively in statistical wave propagation problems (e.g., Tartarskii, 1971; Ishimaru, 1978; Wu and Flatté, 1990) to mathematically simplify the calculation of Transverse Coherence Functions (TCFs), namely, the cross-correlation functions of phase and log-amplitude fluctuations. Such functions are important in determining heterogeneity scales and perturbation strengths (Flatté and Wu, 1988). However, the validity of the delta-correlated assumption has rarely been examined rigorously in the literature. In this paper, we will present a rigorous mathematical derivation on this approximation and its application in the study of TCFs. To illustrate the problem, we first present a brief overview of the Rytov and parabolic approximations, then an exact TCF formulation without the delta-correlated assumption including reinterpretation of the meaning and validity of this assumption, and finally, numerical examples are presented for inverting the spectrum of a stationary random medium.

2. Rytov and Parabolic Approximations

The Rytov approximated solution to the wave equation is the 1\textsuperscript{st} order term in a series arising from the method of smooth perturbation (e.g., Tartarskii, 1971; Rytov et al., 1989). It is a single-scattering theory and does not account for the multiple scattering, which limits it to small velocity perturbations of the medium. The perturbation method (e.g., Wu, 1985) is among the most popular theoretical tools for study of wave propagation in random medium. In this method, the random velocity field $c(\hat{r})$ is decomposed into a constant background velocity $c_0$ plus the remaining perturbation $\delta c(\hat{r})$. The monochromatic wave equation for the wave motion $U$ for acoustics in such a medium with constant density reads,

$$\left[\nabla^2 + k^2\right]U = -2k^2 \delta n(\hat{r}) U,$$

where $k = \omega/c_0$; $\nabla^2$ and $\omega$ are Laplacian differential operator and the angular frequency; \(\delta n(\hat{r}) = (1/2)\left(\frac{c_0}{c^2} - 1\right)\) is the dimensionless velocity fluctuation. Let $U_0$ be the background wavefield without the velocity fluctuation, i.e.,

$$\left[\nabla^2 + k^2\right]U_0 = 0.$$


The background wavefield can be equated to the perturbed wavefield by

\[ U = U_0 e^{\nu}. \]  

(3)

Equations (1,2,3) together yield an integral equation,

\[ \psi(\vec{r}) = \frac{1}{U_0(\vec{r})} \int G(\vec{r};\vec{r}') U_0(\vec{r}') \left[ 2k^2 \delta n(\vec{r}') + \left| \nabla \psi(\vec{r}') \right|^2 \right] d^3 \vec{r}'. \]  

(4)

By assuming \( \left| \nabla \psi(\vec{r}') \right|^2 \ll 2k^2 \delta n(\vec{r}') \), we arrive at the Rylov approximation,

\[ \psi(\vec{r}) = \frac{2k^2}{U_0(\vec{r})} \int G(\vec{r};\vec{r}') U_0(\vec{r}') \delta n(\vec{r}') d^3 \vec{r}'. \]  

(5)

It is widely recognized that the Rylov approximation is superior to the Born approximation in that the Rylov approximation only requires the smoothness of \( \psi(\vec{r}) \) while the Born approximation additionally requires \( \left| \psi(\vec{r}) \right| \ll 1 \). The wavefield can be expressed in terms of amplitude and phase, \( U = Ae^{i\phi} \), \( U_0 = A_0 e^{i\phi_0} \) and \( \psi = \ln(A/A_0) + i(\phi - \phi_0) \). Physically, the Born approximation is valid only in a weak scattering regime and for short propagation distances.

Short-wave propagation in random medium is one of the most extensively studied subjects by many people, partly because it can be approached with simpler theory. When the wavelength \( \lambda \) is large compared to \( a \) the scale of heterogeneity, the scattering is isotropic, but when \( \lambda \) is small, the scattered energy mainly concentrates in a forward cone with angular span \( \sim 1/ka \), \( k \) being the wavenumber (e.g., Sato and Fehler, 1998). Under small-angle approximation suitable for the latter situation, TCFs for two-dimensional and three-dimensional media are the same. We will use the parabolic approximation to simplify equation (5) in two-dimensional case with a vertically incident plane wave,

\[ \psi(x,L) \approx 2k^2 \int_0^L dz' \int \pm_{\infty} dx G(x,L;x',z') \delta n(x',z') \exp\left[ -ik(L-z') \right]. \]  

(6)

The Green’s function in background medium,

\[ G(x,L;x',z) = \frac{i}{4} H_0(kR), R = \sqrt{(x-x')^2 + (L-z)^2}, \]  

where \( H_0(kR) \) denotes the 0th order Hankel function of the first kind. When \( kR \) is large, the asymptotic representation of \( G \) becomes,

\[ G(x,L;x',z') \approx \frac{i}{4} e^{-i\pi/4} \sqrt{\frac{2}{\pi k(L-z')}} \exp\left[ ik(L-z') + \frac{ik(x-x')^2}{2(L-z')} \right]. \]  

(7)

Plugging (7) into equation (6),

\[ \psi(x,L) \approx \frac{ik^2}{2} e^{-i\pi/4} \int_0^L dz' \int_{-\infty}^{\infty} dx \delta n(x',z') \sqrt{\frac{2}{\pi k(L-z')}} \exp\left[ \frac{ik(x-x')^2}{2(L-z')} \right]. \]  

(8)

Make use of the following spectral representation of medium fluctuation

\[ \delta n(x,z) = \frac{1}{2\pi} \int e^{i\kappa x} d\nu(\kappa,z), \]  

(9)

the convolution of equation (8) becomes multiplication in \( \kappa \) domain,

\[ \psi(x,L) = \frac{1}{\pi} \int_0^{\infty} dz' \int e^{i\kappa z} H(\kappa,L-z') d\nu(\kappa,z'), \]  

(10)

where

\[ H(\kappa,L-z') = \frac{ik}{2} \exp\left[ \frac{-i(L-z')\kappa^2}{2k} \right]. \]  

(11)
Strictly speaking, infinite random medium does not have a spectrum in the Fourier sense, because the existence of a Fourier representation requires \( \delta n(x,z) \) be absolute integrable with respect to variable \( x \). Equation (9) should be understood as a stochastic Fourier-Stieltjes integral of spectral representation for a random medium (e.g., Tartarskii, 1971) with \( d\nu(\kappa,z') \) being a spectrum at wavenumber \( \kappa \). For computational purposes, the conventional Fourier spectrum can be used in equation (9) by assuming periodicity of a finite-dimensional random medium.

3. Delta-correlation and TCF Theory

The log-amplitude (logA hereafter) fluctuation can be easily recognized as (see previous section)

\[
u(x,L) = \frac{1}{2} (\psi + \psi') = \frac{1}{\pi} \int_0^L dz' \int e^{i\kappa \xi} H_r(\kappa, L-z') d\nu(\kappa,z'), \tag{11}
\]

where \( H_r(\kappa, L-z') \) is the real part of \( H(\kappa, L-z') \) and * denotes complex conjugate. The coherence of log-A is,

\[
\langle uu' \rangle = \frac{1}{\pi^2} \int_0^\infty dz' \int_0^\infty d\nu(\kappa,z') d\nu(\kappa,z') \langle H_r(\kappa, L-z') H_r(\kappa, L-z') \rangle \tag{12}
\]

where \( < > \) implies ensemble average. The delta-correlation assumption of the medium at different depths states that

\[
W(\kappa, z'-z^*) = \langle d\nu(\kappa, z') d\nu(\kappa, z^*) \rangle = W(\kappa, 0) \delta(z'-z^*). \tag{13}
\]

With this approximation, equation (12) can be calculated in an effortless way. Note that \( W(\kappa, \eta) \) is an even function about \( \eta \).

Our purpose is to examine its validity and we abandon this approximation from now on. Apply trigonometrical identities to equation (12), we get,

\[
H_r(\kappa, L-z') H_r(\kappa, L-z^*) = \frac{k^2}{4} \left[ \cos \left( \frac{z'-z^*}{2k} \right) - \cos \left( \frac{2L-z'-z^*}{2k} \right) \right]. \tag{14}
\]

Substituting identity (14) into equation (12) and making use of coordinate transformation (Figure 1)

\[
\xi = (z'+z^*)/2, \quad \eta = z'-z^*,
\]

equation (12) becomes,

\[
\langle uu' \rangle = \frac{k^2}{8\pi^2} \int_0^\infty d\xi \int_0^\infty d\eta \int d\kappa e^{i\kappa \xi} \left[ \cos \left( \frac{\eta \kappa^2}{2k} \right) - \cos \left( \frac{L-\xi}{k} \kappa^2 \right) \right] W(\kappa, \eta)
\]

\[
= \frac{k^2}{8\pi^2} \int_0^\infty d\xi \int_0^\infty d\eta \int d\kappa e^{i\kappa \xi} \cos \left( \frac{\eta \kappa^2}{2k} \right) W(\kappa, \eta) - \frac{k^2}{8\pi^2} \int_0^\infty d\xi \int_0^\infty d\eta \int d\kappa e^{i\kappa \xi} \cos \left( \frac{L-\xi}{k} \kappa^2 \right) W(\kappa, \eta)
\]

\[
= \frac{k^2}{4\pi^2} \int d\kappa e^{i\kappa \xi} \int_0^\infty d\eta \int d\eta W(\kappa, \eta) \left[ \cos \left( \frac{\eta \kappa^2}{2k} \right) - \cos \left( \frac{L-\xi}{k} \kappa^2 \right) \right] d\xi
\]

\[
= \frac{k^2}{4\pi^2} \int d\kappa e^{i\kappa \xi} \int_0^\infty d\eta \int d\kappa W(\kappa, \eta) \left[ \cos \left( \frac{\eta \kappa^2}{2k} \right) - \cos \left( \frac{L-\xi}{k} \kappa^2 \right) \right] d\xi \tag{15}
\]

\[
+ \frac{k^3}{4\pi^2} \int d\kappa \kappa e^{i\kappa \xi} \int d\eta W(\kappa, \eta) \left[ \sin \left( \frac{\eta \kappa^2}{2k} \right) - \sin \left( \frac{2L-\eta}{2k} \kappa^2 \right) \right] d\xi \tag{I_1}
\]

\[
+ \frac{k^3}{4\pi^2} \int d\kappa \kappa e^{i\kappa \xi} \int d\eta W(\kappa, \eta) \left[ \sin \left( \frac{\eta \kappa^2}{2k} \right) - \sin \left( \frac{2L-\eta}{2k} \kappa^2 \right) \right] d\xi \tag{I_2}.
\]
Assume \( L > \ell \), for \( \phi > \ell \), \( W(\kappa, z) \approx 0 \), we have
\[
I_2 = \frac{k^3}{4\pi^2} \int \frac{d\kappa}{\kappa^2} e^{i\kappa x} \sum_{n=0}^{\infty} \left( \frac{\eta \kappa^2}{2k} \right)^n W(\kappa, \eta) d\eta
\]
\[
- \frac{k^3}{4\pi^2} \int \frac{d\kappa}{\kappa^2} e^{i\kappa x} \left[ \sum_{n=0}^{\infty} \left( \frac{L \kappa^2}{k} \right)^n \cos \left( \frac{\eta \kappa^2}{2k} \right) W(\kappa, \eta) d\eta \right]
\]
\[
+ \frac{k^3}{4\pi^2} \int \frac{d\kappa}{\kappa^2} e^{i\kappa x} \left[ \sum_{n=0}^{\infty} \left( \frac{L \kappa^2}{k} \right)^n \sin \left( \frac{\eta \kappa^2}{2k} \right) W(\kappa, \eta) d\eta \right].
\]

In fact, with no delta-correlation approximation, we expect,
\[
I_2 = \frac{k^3}{8\pi^2} \int \frac{d\kappa}{\kappa^2} e^{i\kappa x} P(\kappa, \kappa^2/2k) + \frac{k^3}{4\pi^2} \int \frac{d\kappa}{\kappa^2} e^{i\kappa x} \left[ \sum_{n=0}^{\infty} \left( \frac{L \kappa^2}{k} \right)^n \eta \cos \left( \frac{\eta \kappa^2}{2k} \right) W(\kappa, \eta) d\eta \right].
\]

The TCF of the logA fluctuation is \( \langle uu \rangle = I_1 + I_2 \); that for phase can be easily found to be \( \langle \phi\phi \rangle = I_1 - I_2 \).

Under the assumption (13), we arrive at,
\[
\langle u^2 \rangle = \frac{Lk^2}{8\pi^2} \int d\kappa e^{i\kappa x} P\left(\kappa, \frac{\kappa^2}{2k}\right) - \frac{k^3}{8\pi^2} \int d\kappa e^{i\kappa x} P\left(\kappa, \frac{\kappa^2}{2k}\right) \int \frac{d\kappa}{\kappa^2} P\left(\kappa, \frac{\kappa^2}{2k}\right),
\]
\[
\langle \phi^2 \rangle = \frac{Lk^2}{8\pi^2} \int d\kappa e^{i\kappa x} P\left(\kappa, \frac{\kappa^2}{2k}\right) + \frac{k^3}{8\pi^2} \int d\kappa e^{i\kappa x} P\left(\kappa, \frac{\kappa^2}{2k}\right) \int \frac{d\kappa}{\kappa^2} P\left(\kappa, \frac{\kappa^2}{2k}\right).
\]

The second term in the right hand side of \( I_1 \) (or \( I_2 \)) is the error introduced by delta-correlation assumption. The error diminishes monotonically for logA and phase fluctuations as the propagation distance \( L \) increases (Figure 2). However, when keeping the propagation distance fixed \( L = 4a \), the errors for phase fluctuations are little influenced by the wavelength (Figure 3). In both cases, the logA TCFs broaden with increasing wavelengths or increasing propagation distances. This demonstrates that heterogeneities having size of the Fresnel zone \( \sqrt{\lambda L} \) are of great importance in logA fluctuations.

Add equations (15) and (16), we get,
\[
\frac{4\pi^2}{Lk^2} \left( \langle u^2 \rangle + \langle \phi^2 \rangle \right) = \int P\left(\kappa, \frac{\kappa^2}{2k}\right) e^{i\kappa x} d\kappa.
\]

Based on equation (17), it is interesting to see,
\[
P\left(\kappa, \frac{\kappa^2}{2k}\right) = \frac{2\pi}{Lk^2} \int \left( \langle u^2 \rangle + \langle \phi^2 \rangle \right) e^{-i\kappa x} dx.
\]

In fact, with no delta-correlation approximation, we expect,
\[
P\left(\kappa, \frac{\kappa^2}{2k}\right) = \frac{2\pi}{Lk^2} \int \left( \langle u^2 \rangle + \langle \phi^2 \rangle \right) e^{-i\kappa x} dx + \frac{2}{L} \int \eta \cos \left( \frac{\eta \kappa^2}{2k} \right) W(\kappa, \eta) d\eta.
\]
It is clear that TCFs only yield partial spectral information of the heterogeneities and the deterministic model can never be achieved by these functions. But they are sufficient to constrain the sizes of the heterogeneities, which is important in studying the Earth’s internal dynamics. Equations (18,19) are the key results presented in this paper. They allow direct inversion for the power spectrum of the random medium. In anisotropic random medium (i.e., the power spectrum of the scatters varies with angle), equation (18) yields partial spectral information of the random media; however, in isotropic random medium, it provides the full Power Spectral Density Function (PSDF) of the random medium. Its shape provides important information on the statistical distribution of the heterogeneities. For a stationary random medium, equation (18) demonstrates the tradeoff between scattering strength $P$ and the random layer thickness $L$. The second term in equation (19) is proportional to the inverse of the propagation distances. Given $\kappa$, the integral is a fixed quantity. When $L$ is large, this term can be dropped. If the term cannot be dropped, recognizing $P$ and $W$ are Fourier transform pairs, iterative method can be employed to solve equation (19).

Next, we use a simple example to show the theory is valid.

4. Numerical examples of spectrum inversion

In the example, we use equation (18) to do the inversion. We consider a model with dimension of 40km-by-40km with a background velocity of 2.5 km/s. A vertically propagating plane wave impinges at the bottom of the model at $z = 0$ km and 10 receiver lines are placed at depths $z_i = i \times 2.42$ km. We synthesize a random medium with a Gaussian PSDF with root-mean-square velocity perturbations of 1% and characteristic scale $a = 0.5$ km. The upgoing plane wave has a Ricker source time function of central frequency $f_0 = 4.5$ Hz. The logA and phase coherence functions are formed at different propagation distances using synthetic seismograms computed by a FD method (Xie, 1988). The FD method has 4th and 2nd order accuracy in spatial and temporary derivatives. Figure 4 displays power spectra estimated from logA/phase measurements at various depths (i.e., propagation distances). logA and phase coherence functions are formed using the method described by Zheng and Wu (2005). The error, $\delta$, introduced by the delta-correlation assumption also can be studied quantitatively and accurately by this method, which is different with the error estimation approach taken by Tartarskii (1971). Past work involved parametrical random medium spectral assumptions (e.g., Gaussian, exponential, etc.) with the appropriate parameters being fitted to the observed logA/phase coherence functions. However, the most important attribute of equation (18) is that we no longer need such assumptions about the medium spectrum, we can invert for the medium spectrum directly based on our observations instead. Equation (18) makes explicit the tradeoff between propagation distance $L$ and the scattering strength of the random medium, which is in $P$. In this theory, with no delta-correlation assumption, the criterion for the minimum propagation distance is quantified as well.

5. Conclusions

The TCF theory, for the first time, has been derived in a mathematical rigorous way without using the delta-correlation approximation. We found heterogeneities of size $\sqrt{\lambda L}$ are of importance in determining the logA fluctuation functions. Our new formulation shows the TCF does have limited vertical resolution, which is different from previous results (i.e., vertical wavenumber be 0). We defined a new observational variable whose Fourier transform directly yields the PSDF of the random medium. This can be used to determine the scale and statistical
distribution of the heterogeneities with minimal model dependence. Our new approach does not require any a priori assumptions on the power spectrum distribution of the random medium.

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References


Figure 1. Mapping from \((z',z'')\) to \((\eta,\xi)\) by coordinate transformation \(\xi = (z' + z'')/2, \eta = z' - z''\).

Figure 2. Exact (short dash) and delta-correlation approximated (solid) TCFs, together with the error term (long dash) as a function of propagation distances \(L\) in isotropic exponential random medium with scale \(a_x = a_z = a = 1\) km. The wavelength is \(a/2\). (a) \(L = 14a\), (b) \(L = 10a\), (c) \(L = 6a\).
Figure 3. Exact (short dash) and delta-correlation approximated (solid) TCFs, together with the error term (long dash) as a function of wavelength $\lambda$ in isotropic exponential random medium with scale $a_x = a_y = a_z = 1$ km. (a) $\lambda = a/8$, (b) $\lambda = a/4$, (c) $\lambda = a/2$, (d) $\lambda = a$. The propagation distance is $L = 6a$. 
Figure 4 Inverted power spectra of the random media at different propagation distances (blue lines) and the true spectrum (black line). Because of the trade-off between the power spectral amplitude and layer thickness, the spectral amplitude in this diagram only has relative meaning.
One-Way and One-Return Approximations (DeWolf Approximation) for Fast Elastic Wave Modeling in Complex Media

Ru-Shan Wu, Xiao-Bi Xie and Xian-Yun Wu

Summary

The De Wolf approximation has been introduced to overcome the limitation of the Born and Rytov approximations in long range forward propagation and backscattering calculations. The De Wolf approximation is a multiple-forescattering-single-backscattering (MFSB) approximation, which can be implemented by using an iterative marching algorithm with a single backscattering calculation for each marching step (a thin-slab). Therefore, it is also called a one-return approximation. The marching algorithm not only updates the incident field step-by-step, in the forward direction, but also the Green’s function when propagating the backscattered waves to the receivers. This distinguishes it from the first order approximation of the asymptotic multiple scattering series, such as the generalized Bremmer series, where the Green’s function is approximated by an asymptotic solution. The De Wolf approximation neglects the reverberations (internal multiples) inside thin-slabs, but can model all the forward scattering phenomena, such as focusing/defocusing, diffraction, refraction, interference, as well as the primary reflections.

In this chapter, renormalized MFSB (multiple-forescattering single-backscattering) equations and the dual-domain expressions for scalar, acoustic and elastic waves are derived by using a unified approach. Two versions of the one-return method (using MFSB approximation) are given: one is the wide-angle (compared to the screen approximation, no small-angle approximation is made in the derivation), dual-domain formulation (thin-slab approximation); the other is the screen approximation. In the screen approximation, which involves a small-angle approximation for the wave-medium interaction, it can be clearly seen that the forward scattered, or transmitted waves are mainly controlled by velocity perturbations; while the backscattered or reflected waves, are mainly controlled by impedance perturbations. Later in this chapter the validity of the thin-slab and screen methods, and the wide-angle capability of the dual-domain implementation are demonstrated by numerical examples. Reflection coefficients of a plane interface, derived from numerical simulations by the wide-angle method, are shown to match the theoretical curves well up to critical angles. The methods are applied to the fast calculation of synthetic seismograms. The results are compared with finite difference (FD) calculations for the elastic French model. For weak heterogeneities (±15% perturbation), good agreement between the two methods verifies the validity of the one-return approach. However, the one-return approach is about 2-3 orders of magnitude faster than the elastic FD algorithm. The other example of application is the modeling of amplitude versus angle (AVA) responses for a complex reservoir with heterogeneous overburdens. In addition to its fast computation speed, the one return method (thin-slab and complex-screen propagators) has some special advantages when applied to the thin-bed and random layer responses.

Keywords: one-way wave equation, generalized screen propagator, seismic wave modeling, AVO.

1. INTRODUCTION

One-way approximation for wave propagation has been introduced and widely used as propagators in forward and inverse problems of scalar, acoustic and elastic waves (e.g., Claerbout, 1970, 1976; Landers and Claerbout, 1972; Flatté and Tappert, 1975; Corones, 1975; Tappert, 1977; McCoy, 1977; Hudson, 1980; Ma, 1982; Wales and McCoy, 1983; Fishman and McCoy, 1984, 1985; Wales, 1986; McCoy and Frazer, 1986; Collins, 1989, 1993; Collins and Wood, 1991; Stoffa et al., 1990; Fisk and McCarter, 1991; Wu and Huang, 1992; Ristow and Ruhl, 1994; Wu, 1994, 1996, 2003; Wu and Xie, 1994; Wu and Jin, 1997; Grimbergen et al., 1998; Stralen et al., 1998; Wild and Hudson, 1998; Thomson, 1999, 2005; De Hoop et al., 2000; Lee et al., 2000; Wild et al., 2000; Wu et al., 2000a,b; Le Rousseau and de Hoop, 2001; Wu and Wu, 2001; Xie and Wu, 2001, 2005; Han and Wu, 2005). The great advantages of one-way propagation methods are the fast speed of computation, often by several orders of magnitudes faster than the full wave finite difference and finite element methods, and the huge saving in internal memory. The successful extension and applications of one-way elastic wave propagation methods has stimulated the research interest in developing
similar theory and techniques for reflected or backscattered wave calculation. There are several approaches in
extending the one-way propagation method to include backscattering and multiple scattering calculations. The
key difference between these approaches is how to define a reference Green’s function for constructing one-
way propagators. The generalized Bremmer series (GBS) approach (Corones, 1975; De Hoop, 1996; Wapenaar,
1996, 1998; van Stralen et al., 1998; Thomson, 1999; Le Rousseau and de Hoop, 2001) adopts an asymptotic
solution of the acoustic or elastic wave equation in the heterogeneous medium as the Green’s function, i.e. the
one-way propagator in the preferred direction. The multiple scattering series is based on the interaction of
Green’s field (incident field) with the medium heterogeneities. The other approach, i.e. the generalized screen
propagator (GSP) approach (Wu, 1994, 1996, 2003; Wu and Xie, 1993, 1994; Wu et al., 1995; Wild and
Hudson, 1998; de Hoop et al., 2000; Wild et al., 2000; Xie et al., 2000; Xie and Wu, 2001, 2005), on the other
hand, does not use asymptotic solutions. Instead, the approach uses the multiple-forward-scattering (MFS)
corrected one-way propagator as the Green’s function. When the backscattered field, calculated at each thin-
slab, is propagated to the backward direction, the same MFS corrected one-way propagator is used. In surface
wave modeling, Friederich et al. (1993) and Friederich (1999) adopted a similar approach of MFS
approximation (See Chapter 2 by Maupin in this book). However, the approach did not apply the MFS
correction to the one-way propagator in the backward direction for the backscattered waves, and therefore did
not take the full advantages of the De Wolf approximation. In section 2, we will compare GBS and GSP
approaches after introducing the De Wolf multiple scattering series and the related approximation. In the rest
of this chapter we will concentrate on the formulation and applications of the generalized screen approach.

In the generalized screen approach, Wu and Huang (1995) introduced a wide-angle modeling method for
backscattered acoustic waves using the multiple-forward-scattering approximation and a phase-screen
propagator. Xie and Wu (1995; 2001) extended the complex screen method to include the calculation of
backscattered elastic waves under the small-angle approximation. Wu (1996) derived a more general theory for
acoustic and elastic waves using the De Wolf approximation, and the theory provided two versions of
a fast implementation of the thin-slab method and a second order improvement for the complex-screen method.
In section 3, the dual-domain thin-slab formulations for the case of scalar, acoustic and elastic media are
derived. In section 4, a fast implementation of the thin-slab propagator is presented with numerical examples.
The excellent agreement between the thin-slab and the elastic FD method in the numerical examples
demonstrates the validity and efficiency of the theory and method. In section 5, the small-angle approximation
is introduced to derive the screen approximation, which is less accurate for wide angle scattering but is more
efficient than the thin-slab method. The validity and potential of the one return approach and the wide-angle
capability for the dual-domain implementation are demonstrated by numerical examples for both thin-slab and
screen methods applying to the calculation of synthetic seismic sections. In section 6, the thin-slab method is
applied to wave field and amplitude versus offset (AVO) modeling in exploration seismology.

2. BORN, RYTOV, DE WOLF APPROXIMATIONS AND MULTIPLE SCATTERING SERIES

The perturbation approach is one of the well-known approaches for wave propagation, scattering and
imaging (see Chapter 9 of Morse and Feshbach, 1953; Chapter 13 of Aki and Richards, 1980; Wu, 1989).
Traditionally, the perturbation method was used only for weakly inhomogeneous media and short propagation
distance. However, recent progress in this direction has led to the development of iterative perturbation
solutions in the form of a one-way marching algorithm for scattering and imaging problems in strongly
heterogeneous media. For a historical review, see section 3.1 of Wu (2003). In this section, we give a
theoretical analysis of the perturbation approach, including the Born, Rytov and De Wolf approximations, as
well as the multiple scattering series. The relatively strong and weak points of the Born and Rytov
approximations are analyzed. Since the Born approximation is a weak scattering approximation, it is not
suitable for large volume or long-range numerical simulations. The Rytov approximation is a smooth
scattering approximation, which works well for long-range small-angle propagation problems, but is not
applicable to large-angle scattering and backscattering. Then, the De Wolf approximation (multiple
forescattering single backscattering, or “one-return approximation”) is introduced to overcome the limitations
of the Born and Rytov approximations in long range forward propagation and backscattering calculations,
which can serve as the theoretical basis of the new dual-domain propagators.
2.1. Born approximation and Rytov approximation: their strong and weak points

For the sake of simplicity, we consider the scalar wave case as an example. The scalar wave equation in inhomogeneous media can be written as

\[ \left( \nabla^2 + \frac{\omega^2}{c^2(r)} \right) u(r) = 0, \]

where \( \omega \) is the circular frequency, \( r \) is the position vector, and \( c(r) \) is wave velocity at \( r \). Define \( c_0 \) as the background velocity of the medium, resulting in

\[ \left( \nabla^2 + k^2 \right) u(r) = -k^2 \varepsilon(r) u(r), \]

where \( k = \omega/c_0 \) is the background wavenumber and

\[ \varepsilon(r) = \frac{c^2(r)}{c^2} - 1 \]

is the perturbation function (dimensionless force). Set

\[ u(r) = u^0(r) + U(r), \]

where \( u^0(r) \) is the unperturbed wave field or “incident wave field” (field in the homogeneous background medium), and \( U(r) \) is the scattered wave field. Substitute (4) into (2) and notice that \( u^0(r) \) satisfies the homogeneous wave equation, resulting in

\[ u(r) = u^0(r) + k^2 \int_V d^3r' g(r; r') \varepsilon(r') u(r'), \]

where \( g(r; r') \) is the Green’s function in the reference (background) medium, and the integral is over the whole volume of medium. This is the Lippmann-Schwinger integral equation. Since the field \( u(r) \) under the integral is the total field which is unknown, equation (5) is not an explicit solution but an integral equation.

2.1.1. Born approximation

Approximating the total field under the integral with the incident field \( u^0(r) \), we obtain the Born Approximation

\[ u(r) = u^0(r) + k^2 \int_V d^3r' g(r; r') \varepsilon(r') u^0(r'). \]

In general, the Born approximation is a weak scattering approximation, which is only valid when the scattered field is much smaller than the incident field. This implies that the heterogeneities are weak and the propagation distance is short. However, the valid regions of the Born approximation are very different for forward scattering than for backscattering. Forward scattering divergence or catastrophe is the weakest point of Born approximation. For simplicity, we use “forescattering” to stand for “forward scattering”. As can be seen from equation (6), the total scattering field is the sum of scattered fields from all parts of the scattering volume. Each contribution is independent from other contributions since the incident field is not updated by the scattering process. In the forward direction, the scattered fields from each part propagate with the same speed as the incident field, so they will be coherently superposed, leading to the linear increase of the total field. The Born approximation does not obey energy conservation. The energy increase will be the fastest in the forward direction, resulting in a catastrophic divergence for long distance propagation. On the contrary, backscattering behaves quite differently from forescattering. Since there is no incident wave in the backward direction, the total observed field is the sum of all the backscattered fields from all the scatterers. However, the size of
coherent stacking for backscattered waves is about $\lambda/4$ because of the two-way travel time difference. Beyond this coherent region, all other contributions will be cancelled out. For this reason, backscattering does not have the catastrophic divergence even when the Born approximation is used. This can be further explained with the spectral responses of heterogeneities to scatterings with different scattering angles.

From the analysis of scattering characteristics, we know that the forescattering is controlled by the d. c. component of the medium spectrum $W(0)$, but the backscattering is determined by the spectral component at spatial frequency $2k$, i.e. $W(2k)$, where $k$ is the background wavenumber (see, Wu and Aki, 1985; Wu, 1989). The d. c. component of the medium spectrum is linearly increasing along the propagation distance, while the contribution from $W(2k)$ is usually much smaller and increases much slower than $W(0)$. The validity condition for the Born approximation is the smallness for the scattered field compared with the incident field. Therefore, the region of validity for the Born approximation for backscattering is much larger than that for forescattering. The other difference between backscattering and forescattering is their responses to different types of heterogeneities. The backscattering is sensitive to the impedance type of heterogeneities, while forescattering mainly responds to velocity type of heterogeneities. Velocity perturbation will produce travel-time or phase change, which can accumulate to quite large values, causing the breakdown of the Born approximation. This kind of phase-change accumulation can be easily handled by the Rytov transformation. This is why the Rytov approximation performs better than the Born approximation for forescattering and has been widely used for long range propagation in the case of forescattering or small-angle scattering dominance.

### 2.1.2. Rytov approximation

Let $u^0(\vec{r})$ be the solution in the absence of perturbations, i.e.,

$$\left(\nabla^2 + k^2\right)u^0 = 0,$$

and the perturbed wave field after interaction with the heterogeneity as $u(\vec{r})$. We normalize $u(\vec{r})$ by the unperturbed field $u^0(\vec{r})$ and express the perturbation of the field by a complex phase perturbation function $\psi(\vec{r})$, i.e.

$$u(\vec{r})/u^0(\vec{r}) = e^{\psi(\vec{r})}.$$  (8)

This is the Rytov Transformation (see Tatarskii, 1971; or Ishimaru, 1978, p.349). $\psi(\vec{r})$ denotes the phase and log-amplitude deviations from the incident field:

$$\psi = \log u - \log u^0 = \log \left[ A/A^0 \right] + i(\phi - \phi^0),$$  (9)

where $\phi$ and $\phi^0$ are phases of perturbed and unperturbed waves. Combining (2), (7) and (8) yields

$$2\nabla u^0 \cdot \nabla \psi + u^0 \nabla^2 \psi = -u^0 \left( \nabla \psi \cdot \nabla \psi + k^2 \varepsilon \right).$$  (10)

The simple identity

$$\nabla^2 (u^0 \psi) = \psi \nabla^2 u^0 + 2 \nabla u^0 \cdot \nabla \psi + u^0 \nabla^2 \psi,$$

together with (7) results in

$$2\nabla u^0 \cdot \nabla \psi + u^0 \nabla^2 \psi = \left( \nabla^2 + k^2 \right) u^0 \psi.$$  (11)

From (10) and (11) we obtain

$$\left( \nabla^2 + k^2 \right) u^0 \psi = -u^0 \left( \nabla \psi \cdot \nabla \psi + k^2 \varepsilon \right).$$  (12)
The solution of (12) can be expressed as an integral equation:

\[ u^0(\vec{r})\psi(\vec{r}) = \int_{\Omega} d^3\vec{r}' \: g(\vec{r}; \vec{r}') u^0(\vec{r}') \left[ \nabla \psi(\vec{r}') \cdot \nabla \psi(\vec{r}) + k^2 \epsilon(\vec{r}') \right], \]  

(13)

where \( g(\vec{r}; \vec{r}') \) is the Green’s function for the background medium, \( u^0(\vec{r}') \) and \( u^0(\vec{r}) \) are the incident field at \( \vec{r}' \) and \( \vec{r} \), respectively.

Equation (13) is a nonlinear (Ricatti) equation. Assuming \( |\nabla \psi \cdot \nabla \psi| \) is small with respect to \( k^2 |\epsilon| \), we can neglect the term \( \nabla \psi \cdot \nabla \psi \) and obtain a solution known as the Rytov approximation:

\[ \psi(\vec{r}) = \frac{k^2}{u^0(\vec{r})} \int_{\Omega} d^3\vec{r}' \: g(\vec{r}; \vec{r}') \epsilon(\vec{r}') u^0(\vec{r}') \]  

(14)

Now, we discuss the relationship between the Rytov and Born approximations, and their strong and weak points, respectively. By expanding \( e^\psi \) into a power series, the scattered field can be written as

\[ u - u^0 = u^0(\epsilon^\psi - 1) = u^0 \psi + \frac{1}{2} u^0 \psi^2 + \cdots. \]  

(15)

When \( \psi \ll 1 \), i.e., the accumulated phase change is less than one radian (corresponding to about one sixth of the wave period), the terms of \( \psi^2 \) and higher terms can be neglected, and

\[ u - u^0 = u^0 \psi = k^2 \int_{\Omega} d^3\vec{r}' \: g(\vec{r}; \vec{r}') \epsilon(\vec{r}') u^0(\vec{r}'), \]  

(16)

which is the Born approximation. This indicates that when \( \psi \ll 1 \), the Rytov approximation reduces to the Born approximation. In the case of large phase-change accumulation, the Born approximation is no longer valid. The Rytov approximation still holds as long as the condition \( \epsilon^\psi \ll k^2 |\epsilon| \) is satisfied.

Let us look at the implication of the condition \( |\nabla \psi \cdot \nabla \psi| \ll k^2 |\epsilon| \) for the Rytov approximation. Assume that the observed total field after wave interacted with the heterogeneities is nearly a plane wave:

\[ u = A e^{i\vec{k}_1 \cdot \vec{r}}, \]

which could be the refracted wave in the forward direction, or the backscattered field, where \( \vec{k}_1 = k\hat{k}_1 \) and \( \hat{k}_1 \) is a unit vector. Since the incident wave is

\[ u^0 = A_0 e^{i\vec{k}_0 \cdot \vec{r}}, \]

the complex phase field \( \psi \) can be written as

\[ \psi = \log(A/A_0) + i\left(\vec{k}_1 - \vec{k}_0\right) \cdot \vec{r}, \]  

(17)

and

\[ \nabla \psi = \nabla \log(A/A_0) + i\left(\vec{k}_1 - \vec{k}_0\right), \]  

(18)

\[ \nabla \psi \cdot \nabla \psi = \left|\nabla \log(A/A_0)\right|^2 - \left|\vec{k}_1 - \vec{k}_0\right|^2 + 2i\left(\vec{k}_1 - \vec{k}_0\right) \cdot \nabla \log(A/A_0). \]  

(19)

Normally wave amplitudes vary much slower than the phases, so the major contribution to \( \nabla \psi \cdot \nabla \psi \) in (19) is from the phase term \( \left|\vec{k}_1 - \vec{k}_0\right|^2 \). Therefore, the condition for Rytov approximation can be approximately
stated as
\[ \left| \vec{k}_f - \vec{k}_o \right|^2 = 4k^2 \sin^2 \frac{\theta}{2} \ll k^2 \epsilon, \] (20)
where \( \theta \) is the scattering angle. Therefore the Rytov approximation is only valid when the scattering angle (deflection angle) is small enough to satisfy
\[ \sin \frac{\theta}{2} \ll \sqrt{\frac{1}{4} \epsilon} = \frac{1}{2} \sqrt{\frac{c_0^2 - c^2(\rho)}{c^2(\rho)}}. \] (21)

This is a point-to-point analysis of the contributions from different terms (for example, the terms in the differential equation (12)). For the integral equation (13), one needs to estimate the integral effects of \( \nabla \psi \cdot \nabla \psi \) and \( k^2 \epsilon \). The heterogeneities need to be smooth enough to guarantee the smallness of the integral of \( \nabla \psi \cdot \nabla \psi \) which is related to scattering angles, in comparison with the total scattering contribution \( k^2 \epsilon \). In any case, the Rytov approximation is totally inappropriate for backscattering. In the exactly backward direction, \( \theta = 180^\circ \) and \( \sin \frac{\theta}{2} = 1 \), inequality (21) is hardly satisfied. Therefore, although not explicitly specified, the Rytov approximation is a kind of small angle approximation. Together with the parabolic approximation, they formed a set of analytical tools widely used for the forward propagation and scattering problems, such as the line-of-sight propagation problem (e.g. Flatté et al., 1979; Ishimaru, 1978; Tatarskii, 1971). The Rytov approximation is also used in modeling transmission fluctuation for seismic array data (Wu and Flatté, 1990), diffraction tomography (Devaney, 1982, 1984; Wu and Toksöz, 1987). Tatarskii (1971, Chapter. 3B) has some discussions on the relation between the Rytov approximation and the parabolic approximation.

2.2. De Wolf approximation

We see the limitations of both the Born and Rytov approximations. Even in weakly inhomogeneous media, we need better tools for wave modeling and imaging for long distance propagation. Higher order terms of the Born series (defined later in this section) may help in some cases. However, for strong scattering media, Born series will either converge very slowly, or become divergent. That is because the Born series is a global interaction series, each term in the series is global in nature. The first term of the Born approximation is a global response, and the higher terms are just global corrections. If the first term has a big error, it will be hard to correct it with higher order terms. One solution to the divergence of the scattering series is the renormalization procedure. Renormalization methods try to split the operations so that the scattering series can be reordered into many sub-series. We hope that some sub-series can be summed up theoretically so that the divergent elements of the series can be removed. The De Wolf approximation splits the scattering potential into forescattering and backscattering parts and renormalizes the incident field and Green’s function into the forward propagated field and forward propagated Green’s function (forward propagator), respectively (De Wolf, 1971, 1985). The forward propagated field \( u_f \) is the sum of an infinite sub-series including all the multiple forescattered fields. The forward propagator \( G_f \) is the sum of a similar sub-series including multiple forescattering corrections to the Green’s function. For backscattering, only the single backscattered field is calculated at each step, and then propagated in the backward direction using the renormalized forward propagator (Green’s function) \( G_f \). The De Wolf approximation is also called the “one-return approximation” (Wu, 1996, 2003; Wu and Huang, 1995; Wu et al., 2000a, b), since it is a multiple-forescattering-single-backscattering (MFSB) approximation. It is also a kind of local Born approximation since the Born approximation applies only locally to the individual thin-slabs. From previous sections we know that the Born approximation works well for backscattering. With the renormalized incident field and Green’s function, the local Born (MFSB) proved to work surprisingly well for many practical applications. The key is to have good forward propagators. Rino (1988) has obtained better approximation than MFSB in the wavenumber domain and pointed out the error of the De Wolf approximation in the calculation of backscattering enhancement. The error (overestimate) is again due to the violation of the energy conservation law by the Born approximation. Even with a forescattering correction, the backscattered energy is still not removed from the forward
propagated waves for local Born approximation. However, for short propagation distances in exploration seismology and some other applications, the errors in reflection amplitudes may not become a serious problem.

In the following, we will adopt an intuitive approach of derivation to see the physical meaning of the approximation. The De Wolf approximation bears some similarity to the Twersky approximation in the case of discrete scatterers (Twersky, 1964; Ishimaru, 1978). The Twersky approximation includes all the multiple scattering, except the reverberations between pairs of scatterers that excludes the paths which connect the two neighboring scatterers more than once. The Twersky approximation has less restrictions and therefore a wider range of applications than the De Wolf approximation. The latter needs to define the split of forward and back scatterings. We define the scattering to the forward hemisphere as forescattering and its complement as backscattering.

The Lippmann-Schwinger equation (5) can be written symbolically as

\[
  u = u^0 + G_0 \varepsilon u ,
\]

(22)

where \( \varepsilon \) is a diagonal operator in space domain, and \( G_0 \) is a nondiagonal integral operator. If the reference medium is homogeneous, \( G_0 \) will be the volume integral with the Green’s function \( g_0(\vec{r};\vec{r}') \) as the kernel. Formally, (22) can be expanded into infinite scattering series (Born series)

\[
  u = u^0 + G_0 \varepsilon u^0 + G_0 \varepsilon G_0 \varepsilon u^0 + \cdots .
\]

(23)

If we split the scattering potential into the forescattering and backscattering parts

\[
  \varepsilon = \varepsilon_f + \varepsilon_b
\]

(24)

and substitute it into (23), we can have all combinations of multiple forescattering and backscattering. We neglect the multiple backscattering (reverberations), i.e., drop all the terms containing two or more backscattering potentials, resulting in a multiple scattering series which contains terms with only one \( \varepsilon_b \).

The general term will look like

\[
  G_0 \varepsilon_f G_0 \varepsilon_f \cdots G_0 \varepsilon_b G_0 \varepsilon_f \cdots G_0 \varepsilon_f u^0 .
\]

(25)

The multiple forescattering on the left side of \( \varepsilon_b \) can be written as

\[
  G_f^m = \left[ G_0 \varepsilon_f \right]^m G_0 ,
\]

(26)

and on its right side,

\[
  u_f^n = \left[ G_0 \varepsilon_f \right]^n u^0 .
\]

(27)

Collecting all the terms of \( G_f^m \) and \( u_f^n \) respectively, we have

\[
  G_f^M = \sum_{m=0}^{M} \left[ G_0 \varepsilon_f \right]^m G_0 ,
\]

\[
  u_f^N = \sum_{n=0}^{N} \left[ G_0 \varepsilon_f \right]^n u^0 .
\]

(28)

Let \( M \) and \( N \) go to infinite, then the renormalized \( G_f \) (forward propagator) and \( u_f \) (forescattering corrected incident field) are:
\( G_f = \sum_{m=0}^{\infty} \left[ G_0 \varepsilon_f \right]^m G_0, \)
\( u_f = \sum_{n=0}^{\infty} \left[ G_0 \varepsilon_f \right]^n u^0, \)

and the De Wolf approximation (in operator form) becomes
\( u = u_f + G_f \varepsilon_b u_f. \)  \( (30) \)

The observed total field \( u \) in (30) is different for different observation geometries. For transmission problems, the backscattering potential does not have any effect under the De Wolf approximation,
\( u_{\text{transmission}} = u_f. \)  \( (31) \)

On the other hand, for reflection measurement, that is, when the observations are in the same level as or behind the source with respect to the propagation direction, there is no \( u_f \) in the total field,
\( u_{\text{reflection}} = G_f \varepsilon_b u_f. \)  \( (32) \)

Write it into integral form, (30) becomes
\( u(\vec{r}) = u_f(\vec{r}) + \int_{\vec{r}'} d^3 \vec{r}' g_f(\vec{r}, \vec{r}') \varepsilon_b(\vec{r}') u_f(\vec{r}'). \)  \( (33) \)

Note that both the incident field and the Green’s function have been renormalized by the multiple forescattering process through the multiple interactions with the forward-scattering potential \( \varepsilon_f. \)

2.3. The De Wolf Series (DWS) of multiple scattering

De Wolf approximation can be considered as the first term of a multiple scattering series: the De Wolf series. After substituting the decomposition (24) into the Born series (23), we rearrange and recompose the scattering series into a series according to the power of the backscattering potential \( \varepsilon_b. \) The first order in \( \varepsilon_b \) will be the De Wolf approximation (one-return approximation). The second order corresponds to the double backscattering (double reflection, double return) term. The higher terms represent the multiple backscattering series. The whole multiple scattering series (23) can be reorganized into
\( u = u_f + G_f \varepsilon_b u_f + G_f \varepsilon_b G_f \varepsilon_b u_f + \cdots \)
\( = \sum_{m=0}^{M} \left[ G_f \varepsilon_b \right]^m u_f. \)

We see that the zero order term is the forward scattering approximated direct wavefield. It is the direct transmitted wave in the real media. The first order term is the De Wolf approximation, which corresponds to the single backscattering signal, or the primary reflections, as called in exploration seismology. This single backscattering signal is different from the Born approximation where the incident field and the Green’s function are both defined in the background medium (a homogeneous medium).

The De Wolf series and generalized Bremmer series.

Here we point out the differences between the De Wolf series (DWS) and the Generalized Bremmer Series (GBS). The original Bremmer series (Bremmer, 1951) is a geometric-optical series for stratified media, which can be considered as a higher order extension to the regular WKBJ solution (the first order term). Later it was generalized to 3D inhomogeneous media, and was named the generalized Bremmer series (Corones, 1975; De Hoop, 1996; Wapenaar, 1996, 1998; van Stralen et al., 1998; Thomson, 1999; Le Rousseau and de Hoop, 2001). The zero order term (the leading term) of the GBS is a high-frequency asymptotic solution (a WKBJ-like solution or Rytov-like solution) (de Hoop, 1996), and used as the Green’s function for deriving the higher order terms. The Green’s function is not updated when calculating the higher order scattering. Therefore, it is
similar to the Born series in a certain sense. Unlike the DWS, which is a series in terms of medium velocity variation, the GBS is in terms of the spatial derivatives of the medium properties (de Hoop, 1996). Because of the asymptotic nature of its Green’s function, the media need to be “smooth” on the scale of the irradiating pulse (De Hoop, 1996, Van Stralen et al., 1998; Thomson, 1999). Some authors used an equivalent medium averaging process to smooth the medium before the application of the method (Stralen et al., 1998).

Wapenaar’s approach (1996, 1998) is to get an asymptotic solution, without averaging the medium, using the flux normalized decomposition of wavefield. Thomson (1999, 2005) included the second term of the asymptotic series to the asymptotic Green’s function to improve the amplitude accuracy. This zero-order term solution for generally inhomogeneous media can be useful for seismic imaging and inversion (e.g. Berkhout, 1982, Berkhout and Wapenaar, 1989). With careful amplitude correction, this type of Green’s function is an energy-flux conserved propagator in any heterogeneous media (including discontinuities), and is called (Wu et al., 2004; Wu and Cao, 2005) the “transparent propagator”. The first term in the GBS (de Hoop, 1996; Wapenaar, 1996) in fact is not the primary reflection from the real media, but a “primary reflection” on the basis of the asymptotic Green’s function. Because the incident field and Green’s function are not updated in deriving the first order term, the “primary reflection” thus obtained is similar to a “distorted primary reflection”, following the term of “distorted Born approximation” in the physics literature. The real primary reflection, which is the first term in the De Wolf series, includes some multiple scattering terms in the GBS.

3. A DUAL-DOMAIN THIN-SLAB FORMULATION FOR ONE-RETURN (MFSB) SYNTHETICS

The physical meaning of the one-return approximation is shown in Figure 1. First a preferred direction needs to be selected for the forward/backward scattering decomposition. In Figure 1 we choose the z-direction as the preferred one. We see that all the solid wave paths have only one backscattering with respect to the z-direction, and therefore will be included in the simulation using the one-return approximation. On the other hand, the dashed line has three backscattering points and cannot be modeled by the one-return approximation.

Figure 2 illustrates schematically the numerical implementation of one-return approximation using a thin-slab marching algorithm. The heterogeneous medium is sliced into numerous thin-slabs. Within each thin-slab, the local Born approximation can be utilized for the calculation of the forward and backward scatterings. The forescattered field is used to update the incident field (the forward propagated wavefield $u_f$) and the backscattered field $u_b$ is stored for later use. This procedure is iteratively done slab by slab, until the end of the model is reached. At the bottom of the model, $u_f$ will be the primary transmitted wave. To get the primary reflected waves, the marching will be done in the reversed direction, i.e. from the bottom to the top of the model. At each thin-slab, the stored backscattered wavefield will be picked up and propagated to the receiving point with the forescattering updated Green’s function $G_f$. In the next few sections, the derivation and formulations of the thin-slab and $u^f$ screen approximations for scalar, acoustic and elastic waves will be given.

3.1. The case of scalar media

Based on the De Wolf approximation (33) and the marching algorithm shown in Figure 2, at each step we need to calculate the fore- and backscattered wave fields caused by the thin-slab. We can choose the thickness of the thin-slab to be thin enough so that the local Born approximation can be applied to the calculation.

Replacing $u^0$ in the Born approximation (6) with the updated local incident field $u^f$, the scalar pressure field at an observation point $x^*$ can be expressed as

$$p(x^*) = p^f(x^*) + k^2 \int d^3x g(x^*,x) F(x) p^f(x)$$

(35)

where $p^f$ is the local incident pressure field and $g$, the Green's function in the thin-slab. $F(x)$ is the perturbation function (scattering potential),

$$F(x) = \frac{c^2}{c^2(x)} - 1 = \frac{s^2(x) - s_0^2}{s_0^2},$$

(36)
with \( s = 1/c \) as the slowness, where \( c \) is the velocity. The second term in the right hand side of (35) is the scattered field and the volume integration is over the thin-slab. Choosing the \( z \)-direction as the main propagation direction, the scattered field at a receiving point \( (\mathbf{x}^*, z^*) \) can be calculated as

\[
P(\mathbf{x}^*, z^*) = k^2 \int d^3 \mathbf{x} g(\mathbf{x}^*; \mathbf{x}) F(\mathbf{x}) p^f(\mathbf{x}),
\]

where \( \mathbf{x}^* \) is the horizontal position in the receiver plane at depth \( z^* \). In the derivations of the next few sections, we use a thin-slab geometry as illustrated in Figure 3. Within the slab the Green’s function is assumed to be a constant medium Green’s function \( g^0 \). Set \( z' \) and \( z_1 \) as the slab entrance (top) and exit (bottom) respectively, and Fourier-transform equation (37) with respect to \( x^* \), resulting in

\[
P(K^*, z^*) = k^2 \int dz g^0(K^*, z^*; x) F(x) p^f(x)
\]

where

\[
g^0(K^*, z^*; x) = \frac{i}{2\gamma} e^{i[z^*-z]K^*_{x^*}}
\]

is the wavenumber-domain Green’s function in constant media (see Berkhout, 1987; Wu, 1994), and

\[
\gamma = \sqrt{k^2 - K^2_{T}}
\]

is the vertical wavenumber (or the propagating wavenumber). Substituting (39) into (38) yields

\[
P(K^*, z^*) = \frac{i}{2\gamma} k^2 \int dz e^{i[z^*-z]K^*_{x^*}} \int d^2 x^* e^{-iK^*_{x^*}} \left[ F(\mathbf{x}^*, z^*) p^f(\mathbf{x}^*, z^*) \right].
\]

Note that the two dimensional inner integral is a 2-D Fourier transform. Therefore, the dual-domain technique can be used to implement (41). If \( x^* = x_1 \), (41) is used to calculate the forward scattered field and update the incident field (transmitted waves) (35); On the other hand, if \( x^* = x^1 \), (41) is for the calculation of backscattered waves. In the case that only forward propagation is concerned, the iterative implementation of (35) and (41) composes a one-way propagator. The derivation of (35) and (41) is based on the local Born approximation, however, the implementation in dual domains is similar to the classical phase-screen approach of a one-way propagator. It is known that the Born approximation is basically a low-frequency approximation, and has severe phase errors for strong contrast and high-frequencies. In order to have better phase accuracy, which is important for imaging (migration), certain high-frequency asymptotic phase-matching has been applied to the local Born solution such that the travel time of the solution match exactly the geometric–optical (ray) travel time in the forward direction. Even a zero-order matching leads to a solution better than the classic phase-screen method, i.e. the spit-step Fourier method (e.g. Stoffa et al, 1990). The method was originally called the "pseudo-screen" method (Wu and De Hoop, 1996; Huang and Wu 1996; Jin et al., 1998, 1999) to distinguish the new form of screen propagator from the classic phase-screen propagator. The phase-screen has operations only in the space domain so that the phase correction is accurate only for small-angle waves; while the pseudo-screen has operations in both the space and wavenumber domains to improve the accuracy for large-angle waves. The asymptotic phase-matching method used by Jin and Wu (1999) and Jin et al. (1998, 1999, 2002) in the hybrid pseudo-screen propagator applies a wavenumber filter in the form of continued fraction expansion and can improve the large-angle wave response significantly. In the method, the wide-angle correction is implemented with an implicit finite difference scheme and the expansion coefficients are optimized by phase-matching. The pseudo-screen propagator belongs to a more general category of generalized screen propagators (GSP) (Wu, 1994, 1996; Xie and Wu, 1998; de Hoop et al., 2000; Le Rousseau
et al., 2001). The hybrid GSP with finite difference wide-angle correction is similar to the Fourier-finite difference (FFD) method (Ristow and Ruhl, 1994; Huang and Fehler, 2000).

3.2. The case of acoustic media

For a linear isotropic acoustic medium, the wave equation in frequency domain is

$$\nabla \cdot \left( \frac{1}{\rho} \nabla p + \frac{\omega^2}{\kappa} p \right) = 0,$$

(42)

where \( p \) is the pressure field, \( \rho \) and \( \kappa \) are the density and bulk module of the medium, respectively. Assuming \( \rho_0 \) and \( \kappa_0 \) as the parameters of the background medium, equation (42) can be written as

$$\frac{1}{\rho_0} \nabla^2 p + \frac{\omega^2}{\kappa_0} p = -\left[ \frac{\omega^2}{\kappa_0} \nabla \left( \frac{1}{\kappa_0} \right) p + \nabla \left( \frac{1}{\rho_0} \right) \nabla p \right],$$

(43)

or

$$(\nabla^2 + k^2) p(x) = -k^2 F(x) p(x),$$

(44)

which is the same as the case of scalar media except

$$F(x) = \delta_\kappa(x) + \frac{1}{k^2} \nabla \cdot \delta_\rho \nabla,$$

(45)

where \( F(x) \) is an operator instead of a scalar function, with

$$\delta_\kappa(x) = \frac{\kappa_0}{\kappa(x)} - 1,$$

(46)

and

$$\delta_\rho(x) = \frac{\rho_0}{\rho(x)} - 1.$$

(47)

If \( \rho \) is kept constant \( (\rho = \rho_0) \), then \( \delta_\kappa = c_0^2/c^2 - 1 \), going back to the scalar medium case.

Following derivation of equation (41), using the thin-slab geometry and the dual-domain expression for the acoustic media, the scattered pressure field at the receiving depth \( z^* \) can be written explicitly as

$$P(K_T,z^*) = \frac{i}{2\rho} \int d^2 x_T e^{ikz^*} \left\{ \int d^2 x_T e^{-iK_T \cdot x_T} \left[ \delta_\kappa(x_T,z) p'(x_T,z) \right] + \frac{i}{k} \int d^2 x_T e^{-iK_T \cdot x_T} \left[ \delta_\rho(x_T,z) \nabla p'(x_T,z) \right] \right\},$$

(48)

where

$$\hat{k} = \frac{1}{k}(K_T,k_z),$$

(49)
and $k_z = \pm \gamma$ for forescattering and backscattering, respectively. The incident field $p^f(x', z)$ and its gradient $\nabla p^f(x', z)$ at depth $z$ can be calculated from the field $p^0(x', z')$ at the slab entrance

$$p^f(x) = p^f(x', z) = \frac{1}{4\pi} \iint d^2 K_{x',z}^f p^0(K_{x',z}^f) e^{i\gamma(x'-x)} e^{ik_{x',z}},$$

and

$$\nabla p^f(x) = \frac{ik}{4\pi} \iint d^2 K_{x',z}^f \hat{k} p^0(K_{x',z}^f) e^{i\gamma(x'-x)} e^{ik_{x',z}},$$

where

$$\hat{k}' = \frac{1}{k}(K_{x',z}^f, \gamma').$$

**Numerical tests for reflected waves in acoustic media**

In the dual-domain thin-slab formulation, no small-angle approximation is made. The only approximation is the smallness of perturbations within each thin-slab so that the background Green's function can be applied and the incident waves can be treated as propagated in the background media. In the limiting case, the thickness of the thin-slab can be shrunk to a one grid step. In that case, the only approximation involved is the dual-domain implementation (or split step algorithm). Numerical tests on the wide-angle version (thin-slab approximation) of the acoustic one-return method showed that the reflection coefficients calculated from the synthetic acoustic records agree well with the theoretical predictions when the incidence angles are smaller than the critical angle in the case of high-velocity layer reflection, and to approximately 70° in the case of low-velocity layer reflection (Wu and Huang, 1992, 1995; Wu, 1996).

### 3.3. The case of elastic media

The equation of motion in a linear, heterogeneous elastic medium can be written as (Aki and Richards, 1980)

$$-\omega^2 \rho(x) u(x) = \nabla \cdot \sigma(x),$$

where $u$ is the displacement vector, $\sigma$ is the stress tensor (dyadic) and $\rho$ is the density of the medium. Here we assume no body force exists in the medium. We know the stress-displacement relation

$$\sigma(x) = c(x) : \varepsilon(x) = \frac{1}{2} c : (\nabla u + u \nabla),$$

where $c$ is the elastic constant tensor of the medium, $\varepsilon$ is the strain field, $u \nabla$ stands for the transpose of $\nabla u$, and “$:$” stands for double scalar product of tensors defined through $(ab) : (cd) = (b \cdot c)(a \cdot d)$. Equation (53) can be written as a wave equation of the displacement field:

$$-\omega^2 \rho(x) u(x) = \nabla \cdot \left[ \frac{1}{2} c : (\nabla u + u \nabla) \right].$$

If the parameters of the elastic medium and the total wave field can be decomposed as
\[ \rho(x) = \rho_0 + \delta \rho(x), \]
\[ \mathbf{c}(x) = \mathbf{c}_0 + \delta \mathbf{c}(x), \]
\[ \mathbf{u}(x) = u^0(x) + U(x), \] (56)

where \( \rho_0 \) and \( \mathbf{c}_0 \) are the parameters of the background medium, \( \delta \rho \) and \( \delta \mathbf{c} \) are the corresponding perturbations, \( u^0 \) is the incident field and \( U \) is the scattered field, then (55) can be rewritten as:
\[ -\omega^2 \rho_0 U - \nabla \cdot \left[ \frac{1}{2} \mathbf{c}_0 : (\nabla U + U \nabla) \right] = \mathbf{F} \] (57)

where
\[ \mathbf{F} = \omega^2 \delta \rho \mathbf{u} + \nabla \cdot [\delta \mathbf{c} : \varepsilon] \]
is the equivalent body force due to scattering.

Similar to (35), we can write the equation of the De Wolf approximation for elastic displacement field as
\[ \mathbf{u}(x_r, z^*) = \mathbf{u}^f(x_r, z^*) + \int d^3x \left\{ \delta \rho \omega^2 \mathbf{u}^f(x_r, z) + \nabla \cdot \left[ \delta \mathbf{c} : \varepsilon^f(x_r, z) \right] \right\} \cdot \mathbf{G}^f(x_r, z^*; x_r, z). \] (58)

Following the derivation of equation (38), we can express the scattered displacement field for a thin-slab in the horizontal wavenumber domain using local Born approximation as
\[ \mathbf{U}(\mathbf{K}_r, z^*) = \int d^3z \int d^2x_r \left\{ \delta \rho \omega^2 \mathbf{u}^f(x_r, z) + \nabla \cdot \left[ \delta \mathbf{c} : \varepsilon^f(x_r, z) \right] \right\} \cdot \mathbf{G}^0(\mathbf{K}_r, z^*; x_r, z) \] (59)

where \( \mathbf{I} \) is the unit dyadic, and
\[ \mathbf{G}^0(z^*, \mathbf{K}_r; z, x_r) = \frac{i k_{\alpha}^2}{2 \rho_0 \omega^2} \hat{k}_\alpha \hat{k}_\alpha \frac{1}{\gamma_\alpha} e^{\hat{k}_\alpha \cdot \mathbf{r}} + \frac{i k_{\beta}^2}{2 \rho_0 \omega^2} (\mathbf{I} - \hat{k}_\beta \hat{k}_\beta) \frac{1}{\gamma_\beta} e^{\hat{k}_\beta \cdot \mathbf{r}}, \] (60)

where \( k_{\alpha} = \omega / \alpha_0 \) and \( k_{\beta} = \omega / \beta_0 \) are P and S wavenumbers with \( \alpha_0 \) and \( \beta_0 \) as the P and S wave background velocities of the thin-slab, respectively. For isotropic media,
\[ \delta \mathbf{c}(\mathbf{x}) : \varepsilon(\mathbf{x}) = \delta \lambda(\mathbf{x}) |\varepsilon| \mathbf{I} + 2 \delta \mu(\mathbf{x}) \varepsilon(\mathbf{x}). \] (62)

Substituting (60) into (59), we can derive the dual-domain expressions for scattered displacement fields in isotropic elastic media.

For P to P scattering:
\[
U^{pp}(K_T, z^*) = \frac{i k_a^2}{2 \gamma_a} \int_{z^*}^z dze^{ik_a(z^*-z)} \times \\
\left\{ \hat{k}_a \hat{k}_a \cdot \iint d^2x_T e^{-iK_T x_T} \frac{\delta \rho(x_T, z)}{\rho} u_a^f(x_T, z) \right. \\
- \hat{k}_a \iint d^2x_T e^{-iK_T x_T} \frac{\delta \mu(x_T, z)}{\lambda + 2 \mu} \nabla \cdot u_a^f(x_T, z) \\
- \hat{k}_a \left( \hat{k}_a \hat{k}_a \right) \cdot \iint d^2x_T e^{-iK_T x_T} \frac{\delta \mu(x_T, z)}{\lambda + 2 \mu} \nabla \cdot u_a^f(x_T, z) \right\} ,
\]

(63)

with \( k_{z}^\alpha = +\gamma_{\alpha} \) for forescattering and \( k_{z}^\alpha = -\gamma_{\alpha} \) for backscattering, and \( \hat{k}_a = (K_T, k_z^\alpha)/k_a \). Note that we replaced \( \rho_0, \lambda_0, \mu_0 \) in denominators by \( \rho = \rho_0 + \delta \rho, \lambda = \lambda_0 + \delta \lambda \) and \( \mu = \mu_0 + \delta \mu \). This replacement is the result of asymptotic matching between the Born approximation for large-angle scattering and the h-f asymptotic travel-time (phase) for forward propagation. It is proved (Wu and Wu, 2003) that with this replacement (asymptotic matching), the phase-shift in the exact forward direction is accurate and the phase error for small angles is reduced compared with the Born approximation. In the meanwhile, the phase error for large angle scattering is much smaller than that of the phase screen approximation.

In (63) \( u_a^f(x_T, z) \), \( \nabla \cdot u_a^f(x_T, z) \) and \( \varepsilon_a^f(x_T, z) \) can be calculated by:

\[
\begin{align*}
\varepsilon_a^f(x_T, z) &= \frac{1}{4\pi} \iint d^2K_T e^{iK_T x_T} u_0^0(K_T) e^{i\phi_a^f(z^-z)} \\
\nabla \cdot u_a^f(x_T, z) &= \frac{1}{4\pi} \iint d^2K_T e^{iK_T x_T} \hat{k}_a \cdot u_0^0(K_T) e^{i\phi_a^f(z^-z)} , \\

\left. \frac{1}{ik_a} \varepsilon_a^f(x_T, z) \right| &= \frac{1}{4\pi} \iint d^2K_T e^{iK_T x_T} \frac{1}{2} \left[ \hat{k}_a \cdot u_0^0(K_T) + \hat{k}_a \cdot u_0^0(K_T) \right] e^{i\phi_a^f(z^-z)} \\
\left. \frac{1}{ik_a} \varepsilon_a^f(x_T, z) \right| &= \frac{1}{4\pi} \iint d^2K_T e^{iK_T x_T} \hat{k}_a \cdot u_0^0(K_T) e^{i\phi_a^f(z^-z)} ,
\end{align*}
\]

(64)

where \( u_0^0(K_T) = |u_0^0(K_T)| \) and \( \hat{k}_a = (K_T, k_z^\alpha)/k_a \).

For P to S scattering:

\[
\begin{align*}
U^{ps}(K_T, z^*) &= \frac{ik_{\beta}^2}{2 \gamma_{\beta}} \int_{z^*}^z dze^{ik_{\beta}(z^*-z)} \left\{ \left( I - \hat{k}_{\beta} \hat{k}_{\beta} \right) \cdot \iint d^2x_T e^{-iK_T x_T} \frac{\delta \rho(x_T, z)}{\rho} u_a^f(x_T, z) \right. \\
- \left( I - \hat{k}_{\beta} \hat{k}_{\beta} \right) \left[ \hat{k}_{\beta} \cdot \iint d^2x_T e^{-iK_T x_T} 2 \frac{\delta \mu(x_T, z)}{\mu} \frac{1}{ik_{\beta}} \varepsilon_a^f(x_T, z) \right] ,
\end{align*}
\]

(65)

where \( \hat{k}_{\beta} = (K_T, k_z^\beta)/k_{\beta} \).

For S to P scattering:
\[
U^{SP}(K_T, z^*) = \frac{i k^2}{2 \gamma} \int_{z}^{z^*} dz e^{i \mathbf{k}_z (z - z')} \left( \mathbf{k}_z \mathbf{k}_z \cdot \left[ \left[ \int d^2 \mathbf{x}_T e^{-i \mathbf{k}_x \cdot \mathbf{x}_y} \frac{\delta \rho(x, z)}{\rho} \right] \mathbf{u}_\beta(x, z) \right] - \left( \mathbf{k}_z \mathbf{k}_z \cdot \left[ \left[ \int d^2 \mathbf{x}_T e^{-i \mathbf{k}_x \cdot \mathbf{x}_y} \frac{1}{\mu} \frac{\delta \mu(x, z)}{i k} \epsilon'_\beta(x, z) \right] \right) \right).
\]

(66)

For S to S scattering:

\[
U^{SS}(K_T, z^*) = \frac{i k^2}{2 \gamma} \int_{z}^{z^*} dz e^{i \mathbf{k}_z (z - z')} \left( (1 - \mathbf{k}_z \mathbf{k}_z) \cdot \left[ \left[ \int d^2 \mathbf{x}_T e^{-i \mathbf{k}_x \cdot \mathbf{x}_y} \frac{\delta \rho(x, z)}{\rho} \right] \mathbf{u}_\beta(x, z) \right] - (1 - \mathbf{k}_z \mathbf{k}_z) \cdot \left[ \left[ \int d^2 \mathbf{x}_T e^{-i \mathbf{k}_x \cdot \mathbf{x}_y} \frac{1}{\mu} \frac{\delta \mu(x, z)}{i k} \epsilon'_\beta(x, z) \right] \right) \right).
\]

(67)

In equations (66) and (67), \( \mathbf{u}_\beta(x, z) \) and \( \epsilon'_\beta(x, z) \) can be calculated by

\[
\mathbf{u}_\beta(x, z) = \frac{1}{4 \pi^2} \int d^2 \mathbf{K}' \epsilon\mathbf{K}' \cdot \mathbf{x} \mathbf{u}_\beta(\mathbf{K}_T) e^{i \mathbf{K}' (z - z')},
\]

\[
\frac{1}{i k} \epsilon'_\beta(x, z) = \frac{1}{4 \pi^2} \int d^2 \mathbf{K}' \epsilon\mathbf{K}' \cdot \mathbf{x} \frac{1}{2} \left[ \mathbf{k}_z \mathbf{k}_z \cdot \left[ \left[ \int d^2 \mathbf{x}_T e^{-i \mathbf{k}_x \cdot \mathbf{x}_y} \frac{1}{\mu} \frac{\delta \mu(x, z)}{i k} \epsilon'_\beta(x, z) \right] \right] \right],
\]

(68)

where \( \mathbf{k}_z = (K_T, \gamma') / \gamma \).

3.4. Implementation procedure of the one-return simulation

Under the MFSB approximation we can update the total field with a marching algorithm in the forward direction. We can slice the whole medium into thin-slabs perpendicular to the propagation direction. Weak scattering condition holds for each thin-slab. For each step forward, the forward and backward scattered fields by a thin-slab between \( z' \) and \( z_i \) are calculated. The forescattered field is added to the incident field so that the updated field becomes the incident field for the next thin-slab. The procedure for acoustic and elastic media can be summarized as follows (see Figure 2). The simplification for the case of scalar media is straightforward.

1. Fourier transform (FT) the incident fields into wavenumber domain at the entrance of each thin-slab.
2. Free propagate in wavenumber domain and calculate the primary fields and its gradients (including strain fields for the case of elastic media) within the slab.
3. Inverse FT these primary fields and its gradients into space domain, and then interact with the medium perturbations: calculation of the distorted fields.
4. FT the distorted fields into wavenumber domain and perform the divergence (and curl, in the case of elastic media) operations to get the backscattered fields. Sum up the scattered fields by all perturbation parameters and multiply it with a weighting factor \( \gamma / 2 \gamma \), then propagate back to the entrance of the slab.
5. Calculate the forescattered field at the slab exit and add to the primary field to form the total field as the incident field at the entrance of the next thin-slab.
6. Continue the procedure iteratively until the bottom of the model is reached.
7. Propagate the backscattered waves from the bottom up to the surface and sum up the contributions of all the thin-slabs during the propagation.

Note that medium-wave interaction, for the case of acoustic waves, involves vector operations and needs 3 pairs of fast Fourier transforms (FFT’s) for each step, while for the case of elastic waves, tensor (strain fields) operations are involved. Due to the symmetric properties of the strain tensors, there are only 6 independent components for each tensor. From (63) - (68) we see that many pairs of FFT’s are required for each step and therefore the computation for elastic wave scattering is rather intensive.

4. FAST ALGORITHM OF THE ELASTIC THIN-SLAB PROPAGATOR AND SOME PRACTICAL ISSUES

4.1. Fast Implementation in dual domains

From equations (63) to (68), we see that the leading-order interactions between incident fields and heterogeneities are expressed in three-dimensional volume integrals. Also the scattered and incident wavenumbers are coupled with each other. So the computation of these equations is still intensive. In this section, the parts of the integration over \( z \) in the equations are analytically estimated. Assume that the slab for each marching step is thin enough that the parameters (velocity and density) can be approximately taken as invariant along \( z \), the integration with respect to \( z \) in equation (63) can be calculated as

\[
\int_{z^2}^{z^1} dz e^{ik_z (z' - z) + i\gamma_a (z' - z^*)} = \Delta z e^{i(\gamma_a + \gamma'_a)\Delta z/2} \text{sinc} \left( \frac{\gamma_a - \gamma'_a}{2} \Delta z \right) \text{ for forescattering } (z^* = z_1),
\]

\[
\Delta z e^{i(\gamma_a + \gamma'_a)\Delta z/2} \sin \left( \frac{\gamma_a + \gamma'_a}{2} \Delta z \right) \text{ for backscattering } (z^* = z').
\]

We see that the integration over \( z \) has been done analytically; however, \( \gamma_a \) and \( \gamma'_a \) are still coupled, which prevents the fast computation of the thin-slab method. To decouple \( \gamma_a \) and \( \gamma'_a \), we neglect the angular variation of amplitude factors but keep the phase information untouched by taking the approximation \( \gamma_a = \gamma'_a = k_a \) for the amplitude factors in equation (69). This assumption is valid for the case where the small-angle scattering is dominant, and therefore the direction of the scattered waves are not far from the incident direction. Under this approximation, equation (69) becomes

\[
\int_{z^2}^{z_1} dz e^{ik_z (z' - z) + i\gamma_a (z' - z^*)} \approx \Delta z e^{i(\gamma_a + \gamma'_a)\Delta z/2} \text{sinc}(k_a \Delta z) \text{ for backscattering } (z^* = z').
\]

For the scattered fields P-S, S-P and S-S, similar approximations can be obtained as follows. For P-S or S-P scattering,
\[
\int_{z_1}^{z_2} dz e^{ik_{\beta}^2(z' - z) + i\gamma_{\alpha}(z - z_0)} \approx \begin{cases} \\
\quad \Delta z e^{i(y_1' + y_\beta)\Delta z/2} \sin[(k_{\alpha} - k_{\beta})\Delta z/2] & \text{for forescattering (} z^* = z_1 \text{),} \\
\Delta z e^{i(y_2' + y_\beta)\Delta z/2} \sin[(k_{\alpha} + k_{\beta})\Delta z/2] & \text{for backscattering (} z^* = z' \text{).}
\end{cases}
\]

For S-S scattering,
\[
\int_{z_1}^{z_2} dz e^{ik_{\beta}^2(z' - z) + i\gamma_{\alpha}(z - z_0)} \approx \begin{cases} \\
\quad \Delta z e^{i(y_1' + y_\beta)\Delta z/2} \sin(k_{\beta}\Delta z) & \text{for forescattering (} z^* = z_1 \text{),} \\
\Delta z e^{i(y_2' + y_\beta)\Delta z/2} \sin(k_{\beta}\Delta z) & \text{for backscattering (} z^* = z' \text{).}
\end{cases}
\]

After integration over \( z \), the integration over transverse plane \( x_T \) in equations (63) – (68) can be carried out by the FFT. In order to further expedite the computation, we can group the scattered field equations (63) to (68) into \( U^P(K_T, z^*) = U^{PP}(K_T, z^*) + U^{SP}(K_T, z^*) \), and \( U^S(K_T, z^*) = U^{PS}(K_T, z^*) + U^{SS}(K_T, z^*) \), i.e.
\[
U^P(K_T, z^*) = \frac{ik_{\alpha}^2}{2\gamma_{\alpha}} e^{i\gamma_{\alpha}\Delta z/2} \Delta z \hat{k}_{\alpha} \left\{ \hat{k}_{\alpha} \int d^2 x_T e^{-ik_{\alpha}x_T} \frac{\delta p(x_T)}{\rho} \left[ \eta^{PP} u_f^f(x_T) + \eta^{SP} u_f^f(x_T) \right] \\
- \int d^2 x_T e^{-ik_{\alpha}x_T} \frac{\delta \rho(x_T)}{\lambda + 2\mu} \hat{k}_{\alpha} \left[ \eta^{PP} u_f^f(x_T) + \eta^{SP} u_f^f(x_T) \right] \right\},
\]
\[
U^S(K_T, z^*) = \frac{ik_{\beta}^2}{2\gamma_{\beta}} e^{i\gamma_{\beta}\Delta z/2} \Delta z (1 - \hat{k}_{\beta} \hat{k}_{\beta}) \left\{ \hat{k}_{\beta} \int d^2 x_T e^{-ik_{\beta}x_T} \frac{2\delta \mu(x_T)}{\lambda + 2\mu} \hat{k}_{\beta} \left[ \eta^{PS} u_f^f(x_T) + \eta^{SS} u_f^f(x_T) \right] \\
- \hat{k}_{\beta} \int d^2 x_T e^{-ik_{\beta}x_T} \frac{2\delta \mu(x_T)}{\mu} \hat{k}_{\beta} \left[ \eta^{PS} u_f^f(x_T) + \eta^{SS} u_f^f(x_T) \right] \right\},
\]

where \( z^* \) (\( z^* = z' \) or \( z^* = z_1 \)) indicates the position of the receiver plane. The modulation factors \( \eta^{PP}, \eta^{SP} = \eta^{PS} \) and \( \eta^{SS} \) are
\[
\eta^{PP} = \begin{cases} \\
1 & \text{for forescattering} \\
\sin(k_{\alpha}\Delta z) & \text{for backscattering}
\end{cases},
\]
\[
\eta^{PS} = \begin{cases} \\
\sin[(k_{\alpha} - k_{\beta})\Delta z/2] & \text{for forescattering} \\
\sin[(k_{\alpha} + k_{\beta})\Delta z/2] & \text{for backscattering}
\end{cases},
\]
\[
\eta^{SS} = \begin{cases} \\
\sin[(k_{\alpha} - k_{\beta})\Delta z/2] & \text{for forescattering} \\
\sin[(k_{\alpha} + k_{\beta})\Delta z/2] & \text{for backscattering}
\end{cases}.
\]
\[ \eta^{SS} = \begin{cases} 1 & \text{for foreshattering} \\ \text{sinc}(k_{\beta} \Delta z) & \text{for backscattering} \end{cases} \] (77)

Note that the factors \( e^{ij_{\gamma}(z-z')} \) and \( e^{i\gamma_{\beta}(z-z')} \) have been replaced by \( e^{i\gamma_{\beta} \Delta z/2} \) and \( e^{ij_{\gamma} \Delta z/2} \) for calculating the background fields. The phase matching (asymptotic matching) has been applied in equations (73) and (74).

Under the above approximations, the thin-slab formulas may be implemented by the procedures as used in the complex screen method (Wu, 1994; Xie and Wu, 2001). First, the whole medium is sliced into appropriate thin-slabs along the overall propagation direction. Weak scattering conditions hold for each thin-slab and the parameters can be considered invariant within each thin-slab in the preferred propagation direction. Suppose that all incident fields, at the entrance of each thin-slab, are given in the wavenumber domain. The implementation procedures may be summarized as follows:

1. Freely propagate in the wavenumber domain and calculate the primary fields, the divergence of incident P wave, and strains (equations 64 and 68).
2. Inverse-FFT the primary fields, the divergence and strains into the space domain, and then calculate the distorted fields (the space domain functions before the FT in equations (73)-(74)).
3. Calculate the foreshattered fields at the thin-slab exit, and add the foreshattered fields to the primary fields to form the total fields as the incident fields for the next thin-slab. In the same time backscattered fields at the thin-slab entrance are also calculated.
4. Continue the procedures 1 to 3 iteratively until the last thin-slab, and the total transmitted fields are obtained at the exit of the last thin-slab.
5. Propagate back the backscattered fields from the last to the first thin-slabs using a similar iterative procedure as step 1 to 3, and sum up all backscattered fields generated by each thin-slab to get the total reflected fields at the surface.

Let us estimate the computation speed. Most calculations involve only fast Fourier transforms. The total computation time can be estimated from the time used in calling FFTs. Taking the complex screen method as a reference, for a 2-D case, the thin-slab method needs 11 inverse FFT’s and 11 forward FFT’s, while the complex screen method needs 5 inverse FFT’s and 7 forward FFT’s. For a 3-D case, the thin-slab method needs 19 inverse FFT’s and 19 forward FFT’s, while the complex screen method needs 7 inverse FFT’s and 10 forward FFT’s. The thin-slab method takes about twice as much time as the complex screen method does, but is still much faster than the full wave methods. We will see a comparison of computation time between the thin-slab method and finite difference method in section 4.4.

### 4.2. Incorporation of boundary transmission/reflection into the one-return method

For the one-return elastic wave propagators mentioned above, the boundary transmission and reflection for a thick-layer (much thicker than the wavelength) is formed by the superposition and interference of numerous thin-slab scatterings. This gives the flexibility of the
thin-slab propagator to treat arbitrarily irregular interface. However, for a homogeneous thick-layer, the calculation of boundary reflection/transmission (R/T) using the reflectivity method (see Aki and Richards, Chapter 5, 1980) is very efficient and accurate. Therefore the reflectivity method has been incorporated into the thin-slab propagator (Wu and Wu, 2003). In addition to the increase of efficiency, this can also reduce the accumulated errors of the thin-slab propagator when propagating in a thick layer with strong contrast of parameters from the surrounding medium. Since the thin-slab method is based on a perturbation approach, the choice of the background medium is an important issue. To make the perturbation small, the background medium parameters are changed as soon as the waves enter a laterally homogeneous region. Take the model shown on Figure 10 as an example. On top of the irregular structure (with grey color) is a homogenous background. When the wave enters the laterally heterogeneous part of the medium, the thin-slab propagator is used for the propagation. We can put in an artificial boundary as shown in the model with bold lines. At that artificial boundary, the reference parameters will jump from the top medium to the grey structure, a -10% jump. The reflection/transmission at that boundary can be calculated using the analytical formulation, such as the Zoeppritz equation, and then the propagation can be done, with an elastic propagator in homogeneous media, to the bottom by one big step. Note that the artificial boundary is added only to facilitate the calculation. The reflected field generated by the reflectivity calculation will be cancelled by the accumulated thin-slab scattered field. Therefore the artificial boundary will not generate spurious arrivals.

The thin-slab propagator is a dual-domain (space-wavenumber domains) approach. In a wavenumber domain, the wavefield can be expressed by a superposition of plane waves. Therefore, incorporating a reflectivity calculation into the thin-slab propagator is straightforward. After plane wave decomposition, the spectra of incident fields can be decomposed into P- and SV- and SH-components, and the R/T formulation can be applied directly to these components. In section 4.4 numerical examples will be given to show the validity and efficiency of this approach.

4.3. Treatment of anelasticity: the Q-factor

Spatially varying quality factors ($Q_p$ and $Q_s$) can also be incorporated into the thin-slab propagator to study the effects of intrinsic attenuation in visco-elastic media. Spatially varying intrinsic attenuation can cause not only wavefield attenuation but also scattering and frequency-dependent reflections. The thin-slab propagator, with spatially varying Q-factor, is an efficient tool for such study.

For elastic, isotropic media, Lamé constants $\lambda$, $\mu$, $\lambda_0$, and $\mu_0$ are related with elastic parameters by

$$\lambda = \rho \alpha^2 - 2 \rho \beta^2, \quad \mu = \rho \beta^2, \quad \lambda_0 = \rho_0 \alpha_0^2 - 2 \rho_0 \beta_0^2, \quad \mu_0 = \rho_0 \beta_0^2,$$

(78)  

(79)
where $\alpha_0$, and $\beta_0$ are compressional- and shear-wave velocities of background medium. We introduce complex velocities by performing the following transforms:

$$\alpha \rightarrow \alpha (1 - i/2Q_P), \quad \beta \rightarrow \beta (1 - i/2Q_S), \quad (80)$$

$$\alpha_0 \rightarrow \alpha_0 (1 - i/2Q_P^0), \quad \beta_0 \rightarrow \beta_0 (1 - i/2Q_S^0), \quad (81)$$

where $Q_P^0$ and $Q_S^0$ are compressional- and shear-wave quality factors of background medium. $i$ is the imaginary unit. Once all parameters in equations (80)-(81) are known, we can calculate the perturbations $\delta \lambda$ and $\delta \mu$ using equations (78)-(79). Now $\lambda_0$ and $\mu_0$ become complex. As a result, the reference wavenumbers of P- and S-waves also become complex. The heterogeneities of quality factors ($Q_P$ and $Q_S$) have been included in the complex $\delta \lambda$ and $\delta \mu$. With the above extension, the dual-domain thin-slab propagators can be used to model visco-elastic seismic responses. In section 6.3 some numerical examples will be given.

### 4.4. Numerical tests for the elastic thin-slab method

#### 4.4.1. Reflection coefficient calculations

The following examples show the angle dependence of amplitudes (reflection coefficients) calculated by the thin-slab method. The model used is defined on a $2048 \times 200$ rectangular grid. The grid spacing in the horizontal direction is 16 m and in the vertical direction is 4 m. A horizontal interface is introduced in the middle of the model. The upper layer has $\alpha = 3.6 \text{km/s}$, $\beta = 2.08 \text{km/s}$ and $\rho = 2 \text{g/cm}^3$, which is taken as background medium. The lower layer has different P and S wave velocities relative to the upper layer. A 15 Hz plane P wave (or S wave) is incident on the interface at different angles. To enhance the stability in the calculation of reflection coefficients, we chose 500 samples (displacement amplitudes in the space domain) at the center of the model for both the incident and reflected fields and calculate their means respectively. Reflection/conversion coefficients are defined by the ratio of the reflected amplitudes to the incident amplitudes at the same receiver plane. Figure 4 shows the reflection/conversion coefficients versus angle of incidence with a perturbation of 10% for both P and S wave velocities. The theoretical reflection/conversion coefficients (dashed lines) are also given as references. The upper panel corresponds to P wave incidence and the lower panel, S wave incidence. The angles of incidence of S wave are limited to the critical angle of S-P converted wave. Both results agree well with the theory for a wide range of incident angles ($55^\circ$ for P wave incidence, and near critical angle $32^\circ$ for S wave incidence).

Figures 5 to 7 show similar results as shown in Figure 4 but with different perturbations of -10%, 20% and -20%, respectively. Figure 5 corresponds to a negative velocity perturbation. For P wave incidence, no critical angle exists. However, errors occur for large angles of incidence ($>65^\circ$ for P wave incidence). This is limited by the ability of the one-way propagator to handle wide-angle waves. For S wave incidence, both results are in good agreement up to the critical angle. Figure 6 corresponds to a perturbation of 20% for both P and S wave velocities. For a small angle of incidence (40° for P wave incidence and 20° for S wave incidence), the thin-slab results match the theoretical values. Comparing Figures 6 and 4, we see that the wide-angle capacity of the thin-slab method decreases as perturbations increase. Figure 7 corresponds to a perturbation of -20% for both P and S wave velocities.
4.4.2. 2D synthetic seismograms

Synthetic seismograms are also generated to further demonstrate the capability of the thin-slab method. Figure 8 shows a 2-D model that is a vertical slice cut from the elastic French model (French, 1974). The model has a strong irregular interface that will generate large-angle reflections and scattering. The parameters of the background medium are taken as $\alpha_0 = 3.6 \text{ km/s}$, $\beta_0 = 2.08 \text{ km/s}$ and $\rho_0 = 2.2 \text{ g/cm}^3$. The layer in black color has a perturbation of -20% for both P and S wave velocities. Source and receiver geometry are also shown in the figure. A Ricker wavelet with a dominant frequency of 20Hz is used. For the thin-slab method, the spacing interval is 8 m in both horizontal and vertical directions. For the finite difference method, the spacing interval is 4 m and time interval is 0.5 ms. The stability criterion is satisfied. The direct arrivals have been properly removed from the finite difference results. Figure 9 shows a comparison of the synthetic seismograms between the thin-slab method and finite difference method. Both amplitudes and arrival times are in good agreement up to large offsets (~1.4 km) compared to the depth (~1 km). For this example, the thin-slab method is about 57 times faster than the finite difference method. The factor will increase as the size of model increases, especially for 3D cases.

4.4.3. 3D synthetic seismograms

Figure 10 shows a 3D French model. The parameters of the background medium are taken as $\alpha_0 = 3.6 \text{ km/s}$, $\beta_0 = 2.08 \text{ km/s}$ and $\rho_0 = 2.2 \text{ g/cm}^3$. The Grey structure has a perturbation of -10% for both P- and S-wave velocities. A Ricker wavelet with a dominant frequency of 10 Hz is used. For the 8th-order 3D elastic finite-difference method (Yoon, 1996), the spacing interval is 20 m. The actual grid size used is $250 \times 250 \times 250$ including 25 grids of absorbing boundary for each face of the model. Time interval used is 0.001 sec and 2500 time steps are calculated. It took about 28 hours. For thin-slab method, the spacing interval used is 20 m in transversal plane, which is the same as used for the finite-difference method. But a fine grid size of 5 m is used in propagation direction. We did the same calculation on the same machine using the thin-slab propagator. It took 2.7 hours. Thin-slab is about 10 times faster than the finite-difference method. Figure 11 gives the 3 components of the synthetic seismograms calculated using the finite-difference method (solid) and by the thin-slab method (dotted). The Y-component in Figure 11 has been multiplied by a factor of 3. We see that the results of the two methods are in excellent agreements for small to mild angle scatterings. Especially for the reflected/converted waves generated at the lower interface of the model, the use of reflectivity method improved the accuracy of the arrival times of those events, and matched well with those by the finite-difference method.

5. THE SCREEN APPROXIMATION

For some special applications, such as slowly varying media, the synthetics only involve small-angle scattering. In this case the screen approximation can be applied to accelerate the computation.

Under small-angle scattering approximation, we can compress the thin-slab into an equivalent screen and therefore change the 3-D spectrum into a 2-D spectrum. Dual-domain implementation of the screen approximation will make the modeling of backscattering very efficient. In the following, the cases of acoustic and elastic media will be given respectively.

5.1. Screen Propagators for acoustic media
5.1.1. Thin-slab formulation in wavenumber domain

In order to further accelerate the computation, approximations to the interaction between the thin-slab and incident waves can be applied. First, we discuss the thin-slab formulation in wavenumber domain, and in the next section, the screen approximation will be made based on the wavenumber domain formulation.

To obtain the wavenumber domain formulation, we carry out analytically the integration along z-direction between the slab entrance \( z' \) and the exit \( z \) (see Figure 3). In the case of acoustic media, we substitute (50) and (51) into equation (48) and perform the moving frame coordinate transform \( \mathbf{z} \to \mathbf{z} - \mathbf{z}' \), resulting in

\[
P(\mathbf{K}_T, z^*) = \frac{i k^2}{2\gamma 4\pi^2} e^{i(z^* - z)} \int d\mathbf{K}'_T \times 
\begin{align}
&\left\{ \int_{0}^{\Delta z} dz \int d^2 \mathbf{x}_T \varepsilon_x (z, \mathbf{x}_T) p^0 (\mathbf{K}'_T) e^{-i(\pm\gamma - \gamma') z} e^{-i(\mathbf{K}'_T - \mathbf{K}_T') \cdot \mathbf{x}_T} \\
&- (\mathbf{k} \cdot \mathbf{k'}) \int_{0}^{\Delta z} dz \int d^2 \mathbf{x}_T \varepsilon_\rho (z, \mathbf{x}_T) p^0 (\mathbf{K}'_T) e^{-i(\pm\gamma - \gamma') z} e^{-i(\mathbf{K}'_T - \mathbf{K}_T') \cdot \mathbf{x}_T} \right\},
\end{align}
\]

where

\[
p^0 (\mathbf{K}'_T) = p^f (z', \mathbf{K}_T)
\]

is the incident field at the slab entrance, \( \Delta z = z_i - z' \), and \( \pm\gamma \) correspond to forescattering and backscattering respectively. Note that

\[
\int_{0}^{\Delta z} dz \int d^2 \mathbf{x}_T \mathcal{F}(\mathbf{x}) e^{-i(\mathbf{k} - \mathbf{k'}) \cdot \mathbf{x}} = \mathcal{F}(\mathbf{k} - \mathbf{k'}),
\]

where \( \mathcal{F}(\mathbf{k}) \) is the 3D Fourier transform of the thin-slab, i.e. the Slab-Spectrum, \( \mathbf{k}' \) is the incident wavenumber (52), and \( \mathbf{k} \) is the outgoing wavenumber (scattering wavenumber) defined as

\[
\mathbf{k} = \mathbf{k}' = \mathbf{K}_T + \gamma \hat{\mathbf{e}}_z
\]

for forescattered field and

\[
\mathbf{k} = \mathbf{k}^b = \mathbf{K}_T - \gamma \hat{\mathbf{e}}_z
\]

for backscattered field. Therefore the Local Born scattering in wavenumber domain can be written as

\[
P(\mathbf{K}_T, z^*) = \frac{i k^2}{8\pi^2 \gamma} e^{ik_z(z^* - z)} \times 
\begin{align}
&\left[ \varepsilon_x (\mathbf{K}_T - \mathbf{K}'_T, k_z - \gamma') - (\mathbf{k} \cdot \mathbf{k'}) \varepsilon_\rho (\mathbf{K}_T - \mathbf{K}'_T, k_z - \gamma') \right] p^0 (\mathbf{K}'_T),
\end{align}
\]

with \( k_z = \gamma \) for forescattering and \( k_z = -\gamma \) for backscattering. When the receiving level is at the bottom of the thin-slab (forescattering), \( z^* = z_i \); while \( z^* = z' \) is for the backscattered field at the
entrance of the thin-slab. The total transmitted field at the slab exit can be calculated as the sum of the foreshattered field and the primary field, which can be approximated as

$$p^0(x',z^*) = \frac{1}{4\pi^2} \int dK_T p^0(K_T) e^{i(x'-x')} e^{iK_Tz'}. \quad (88)$$

We see that the scattering characteristics depend on the spectral properties of heterogeneities. In the case of large-scale heterogeneities, where lateral sizes of heterogeneities are much larger than wavelength, the major foreshattered energy is concentrated in a small cone towards the forward direction. For foreshattering, the outgoing wavenumbers $k$ are nearly in the same direction as the incoming wavenumbers $k'$. Within the small cone, $k - k'$ stays small. Therefore, the scattered waves are controlled by the low lateral spatial-frequency components of heterogeneities. In the meanwhile, $k$ and $k'$ have opposite directions for backscattered waves and the backscattering is most sensitive to those vertical spectral components which are comparable to the half-wavelength (see Wu and Aki, 1985). Also, we know that foreshattering is controlled by velocity heterogeneities, while backscattering responses mostly to impedance heterogeneities (ibid). This point will be seen much more clearly in the next section.

Note that the wave-slab interaction in wavenumber domain [equation (87)] is not a convolution and therefore the operation in space domain is not local. Therefore, the wave-slab interaction in wavenumber domain involves matrix multiplication and is computationally intensive.

5.1.2. Small-angle approximation and the screen propagators

Under this approximation, both incoming and outgoing wavenumbers have small transversal components $K_T$ compared to the longitudinal component $\gamma$ and therefore

$$\gamma = \sqrt{k^2 - K_T^2} \approx k(1 - K_T^2/2k). \quad (89)$$

Then

$$k' - k' = (K_T - K_T') + (\gamma - \gamma')\hat{e}_z$$

$$\approx (K_T - K_T') + (K_T^2 - K_T'^2)/(2k)\hat{e}_z$$

$$\approx (K_T - K_T') + 2k\hat{e}_z$$

and

$$k - k' = (K_T - K_T') - (\gamma + \gamma')\hat{e}_z \approx (K_T - K_T') - 2k\hat{e}_z. \quad (91)$$

5.1.3. Screen approximation in acoustic media

With the small-angle approximation, the 3D thin-slab spectrum (84) can be approximated by:

$$\widetilde{F}(k' - k) \approx \widetilde{F}(K_T - K_T', K_z = 0)$$

$$= \int d^2x\int_0^{\lambda_e} dz [\varepsilon_\rho(x_T,z) - \varepsilon_\rho(x_T,z)]$$

$$= 2\int d^2x e^{i(\omega x + k z)} S_\rho(x_T), \quad (92)$$

$$= \int d^2x e^{i(\omega x + k z)} S_\rho(x_T), \quad (92)$$
where $S_r$ is a 2D screen of velocity perturbation, and

$$\tilde{F}(k^b - k) \approx \tilde{F}(K_r - K'_r, K_z = -2k)$$

$$= \int d^2x_r e^{-i(k_r - k_r')x_r} \int_0^{\Lambda z} dz e^{i2kz} \left[ \epsilon^x(x_r, z) + \epsilon^\rho(x_r, z) \right]$$

$$= 2 \int d^2x_r e^{-i(k_r - k_r')x_r} S_r(x_r),$$

where $S_t$ is a 2D screen of impedance perturbation. We see that with the above approximation, 3-D thin-slab spectra have been replaced by 2-D screen spectra that are slices of the 3-D spectra. In the case of forescattering, the slice is from a velocity spectrum at $K_z = 0$, where $K_z$ is the spatial frequency in the z-direction; while for backscattering, from an impedance spectrum, a slice at $K_z = -2k$. In the special case when $F(x_r, z)$ varies very little along $z$ within the thin-slab, the screen spectra can be further approximated as

$$\tilde{F}(k^f - k) \approx 2 S_r(K_r - K'_r)$$

$$= [\delta^x(K_r - K'_r) - \delta^\rho(K_r - K'_r)]\Delta z,$$

$$\tilde{F}(k^b - k) \approx 2 S_t(K_r - K'_r)$$

$$= [\delta^x(K_r - K'_r) + \delta^\rho(K_r - K'_r)]\Delta z \text{sinc}(k\Delta z)e^{ik\Delta z}.$$ (94)

The scattered fields (74) under the screen approximation become

$$P(K_r, z^r) \approx i \frac{k^2}{4\pi^2} e^{ik(z^r - z)} \int dK_r \tilde{S}(K_r - K'_r) p^0(K'_r).$$ (95)

The above equation is a convolution integral in wavenumber domain and the corresponding operation in space domain is a local one. The dual-domain technique can be used to speed up the computation:

$$P(K_r, z^r) \approx i \frac{k^2}{\gamma} e^{ik(z^r - z)} \int dK_r e^{-iK_r x_r} S(x_r) p^0(x_r),$$ (96)

where

$$S(x_r) = S_r(x_r) = \frac{1}{4} \int_0^{\Lambda z} dz [\delta^x(x_r, z) - \delta^\rho(x_r, z)] \text{ for forescattering},$$ (97)

$$S(x_r) = S_t(x_r) = \frac{1}{4} \int_0^{\Lambda z} dz e^{i2kz} [\delta^x(x_r, z) + \delta^\rho(x_r, z)] \text{ for backscattering}. $$ (98)

The total transmitted field at $z_1$ is

$$P^f(K_r, z_1) = p^0(K_r, z_1) + P^f(K_r, z_1)$$

$$= e^{ik(z_1 - z)} \int dK_r e^{-iK_r x_r} p^0(x_r)[1 + ikS_r(x_r)]$$

$$= e^{ik(z_1 - z)} \int dK_r e^{-iK_r x_r} p^0(x_r) \exp[ikS_r(x_r)].$$ (99)
where \( k/\gamma_a \approx 1 \) has been used for the scattered field based on the small-angle scattering approximation. The above equation is the dual-domain implementation of phase-screen propagation.

5.1.4. Procedure of MFSB using the screen approximation

1. Fourier transform the incident field at the starting plane into wavenumber domain and propagate to the screen using a constant velocity propagator.

2. Inverse Fourier transform the incident field into space domain. Interact with the impedance screen (complex-screen) to get the backscattered field and interact with the velocity screen (phase-screen) to get the transmitted field.

3. Fourier transform the transmitted field into wavenumber domain and propagate to the next screen using a constant velocity propagator.

4. Repeat the propagation and interaction screen-by-screen to the bottom of the model space.

5. Backpropagate and stack the stored backscattered field screen by screen from the bottom to the top to get the total backscattered field on the surface.

Similar to the derivation for acoustic media, the wavenumber domain formulation can be obtained by substituting equations (64), (68) into equations (63) and (65)-(67).

\[
U^{PP}(K_T, K'_T) = \frac{i}{2\gamma_a} k^2_a u^p_0 \hat{k}_a \left[ (\hat{k}_a, \hat{k}'_a) \frac{\delta \rho(\hat{k})}{\rho} - \frac{\delta \lambda(\hat{k})}{\lambda + 2\mu} - (\hat{k}_a \cdot \hat{k}'_a)^2 \frac{2\delta \mu(\hat{k})}{\lambda + 2\mu} \right], \\
U^{PS}(K_T, K'_T) = \frac{i}{2\gamma_\beta} k^2_\beta (1 - \hat{k}_\beta \hat{k}'_\beta) \cdot u^p_0 \left[ \frac{\delta \rho(\hat{k})}{\rho} - \frac{2\beta_\beta}{\alpha_0} (\hat{k}_\beta \cdot \hat{k}'_\beta) \frac{\delta \mu(\hat{k})}{\mu} \right], \\
U^{SP}(K_T, K'_T) = \frac{i}{2\gamma_a} k^2_a (u^s_0, \hat{k}_a) \hat{k}_a \left[ \frac{\delta \rho(\hat{k})}{\rho} - \frac{2\beta_\beta}{\alpha_0} (\hat{k}_\beta \cdot \hat{k}'_\beta) \frac{\delta \mu(\hat{k})}{\mu} \right], \\
U^{SS}(K_T, K'_T) = \frac{i}{2\gamma_\beta} k^2_\beta (1 - \hat{k}_\beta \hat{k}'_\beta) \cdot \left\{ u^s_0 \left[ \frac{\delta \rho(\hat{k})}{\rho} - (\hat{k}_\beta \cdot \hat{k}'_\beta) \frac{\delta \mu(\hat{k})}{\mu} \right] \right\},
\]

(100)

where \( u^p_0 \) is the spectral field of the incident P wave, and \( \delta \rho(\hat{k}), \delta \lambda(\hat{k}) \) and \( \delta \mu(\hat{k}) \) are the three-dimensional Fourier transforms of medium perturbations, and \( \hat{k} = \hat{k} - \hat{k}' \) is the exchange wavenumber with \( \hat{k}' \) and \( \hat{k} \) as incident and scattering wavenumber vectors, respectively.

From the thin-slab formulation, under the small-angle approximation, both incident and scattered wavenumbers have small lateral components \( K_T \) and \( K'_T \) compared to vertical components and therefore

\[
\gamma_a \approx k_a (1 - K^2_k/2k^2_a), \quad \gamma'_a \approx k'_a (1 - K'^2_k/2k'^2_a), \quad \gamma_\beta \approx k_\beta (1 - K^2_\beta/2k^2_\beta), \quad \gamma'_\beta \approx k'_\beta (1 - K'^2_\beta/2k'^2_\beta),
\]

(101)
Using these approximations and neglecting higher-order (greater than second order) small quantities, the scattering patterns can be obtained as

\[
\begin{align*}
\langle \hat{k}_a, \hat{k}_a' \rangle &\approx \left\{ 
\begin{array}{ll}
+1 + \left( |\mathbf{K}_r - \mathbf{K}_r'| \right)^2 / 2k_a^2 & \text{for forescattering} \\
-1 + \left( |\mathbf{K}_r + \mathbf{K}_r'| \right)^2 / 2k_a^2 & \text{for backscattering}
\end{array}
\right. , \\
\langle \hat{k}_a, \hat{k}_\beta' \rangle &\approx \left\{ 
\begin{array}{ll}
+1 - \left( |k_\beta \mathbf{K}_r - k_\alpha \mathbf{K}_r'| \right)^2 / (2k_a^2k_\beta^2) & \text{for forescattering} \\
-1 + \left( |k_\beta \mathbf{K}_r + k_\alpha \mathbf{K}_r'| \right)^2 / (2k_a^2k_\beta^2) & \text{for backscattering}
\end{array}
\right. , \\
\langle \hat{k}_\beta, \hat{k}_a' \rangle &\approx \left\{ 
\begin{array}{ll}
+1 - \left( |k_\alpha \mathbf{K}_r - k_\beta \mathbf{K}_r'| \right)^2 / (2k_a^2k_\beta^2) & \text{for forescattering} \\
-1 + \left( |k_\alpha \mathbf{K}_r + k_\beta \mathbf{K}_r'| \right)^2 / (2k_a^2k_\beta^2) & \text{for backscattering}
\end{array}
\right. , \\
\langle \hat{k}_\beta, \hat{k}_\beta' \rangle &\approx \left\{ 
\begin{array}{ll}
+1 + \left( |\mathbf{K}_r - \mathbf{K}_r'| \right)^2 / 2k_\beta^2 & \text{for forescattering} \\
-1 + \left( |\mathbf{K}_r + \mathbf{K}_r'| \right)^2 / 2k_\beta^2 & \text{for backscattering}
\end{array}
\right. ,
\end{align*}
\tag{102}
\]

To decouple the incident wavenumber \( \mathbf{K}_r \) and the scattered wavenumber \( \mathbf{K}_r' \) in equations (102), suppose that the medium heterogeneities are smooth enough that the scattering wavenumbers \( \hat{k}_a \) or \( \hat{k}_\beta \) deviates not too far from the incident wavenumbers \( \hat{k}'_a \) or \( \hat{k}'_\beta \). We can take \( \mathbf{K}_r' \) to be approximately equal to \( \mathbf{K}_r \), the wavenumber couplings may be simplified further by

\[
\begin{align*}
\langle \hat{k}_a, \hat{k}_a' \rangle &\approx \left\{ 
\begin{array}{ll}
+1 & \text{for forescattering} \\
-1 + 2K_r^2/k_a^2 & \text{for backscattering}
\end{array}
\right. , \\
\langle \hat{k}_a, \hat{k}_\beta' \rangle &\approx \left\{ 
\begin{array}{ll}
+1 - \left( K_r/k_\beta - K_r/k_\alpha \right)^2 / 2 & \text{for forescattering} \\
-1 + \left( K_r/k_\beta + K_r/k_\alpha \right)^2 / 2 & \text{for backscattering}
\end{array}
\right. , \\
\langle \hat{k}_\beta, \hat{k}_a' \rangle &\approx \left\{ 
\begin{array}{ll}
+1 - \left( K_r/k_\alpha - K_r/k_\beta \right)^2 / 2 & \text{for forescattering} \\
-1 + \left( K_r/k_\alpha + K_r/k_\beta \right)^2 / 2 & \text{for backscattering}
\end{array}
\right. , \\
\langle \hat{k}_\beta, \hat{k}_\beta' \rangle &\approx \left\{ 
\begin{array}{ll}
+1 & \text{for forescattering} \\
-1 + 2K_r^2/k_\beta^2 & \text{for backscattering}
\end{array}
\right. .
\end{align*}
\tag{103}
\]

It can be seen from equations (102)-(103) that the first-order corrections have relatively stronger effects on the backscattering than on the forescattering. The scattering pattern \( (\mathbf{I} - \hat{k}_\beta \hat{k}_\beta') \cdot \hat{k}_\beta' \) appearing in equation (103) is zero for forward scattering, while for backward backscattering it is

\[
\left[ (\mathbf{I} - \hat{k}_\beta \hat{k}_\beta') \cdot \hat{k}_\beta' \right]_{\text{backward}} = \left[ \hat{k}_\beta' - (\hat{k}_\beta \cdot \hat{k}_\beta') \hat{k}_\beta \right]_{\text{backward}} \\
\approx \left[ \hat{k}_\beta' - (-1 + 2K_r^2/k_\beta^2) \hat{k}_\beta \right]_{\text{backward}} \\
\approx \left( 2K_r/k_\beta, 2K_r^2/k_\beta^2 \right),
\tag{104}
\]
up to the first-order correction.

5.3. Complex screen method with first-order corrections

Numerical tests show that the effect of the first-order corrections of wave couplings for P-S and S-P forescattering can be neglected. So the first-order corrections are introduced only for backward propagators. Substituting equations (103)-(104) into equation (100) and integrating over incident wavenumber $\hat{k}'$, we can obtain the following formulas

$$
U^p_f(K', K_T) = -ik_\alpha \Delta \tilde{k}_a u_0^p(K_T) \frac{\delta \alpha(\tilde{K}_T)}{\alpha_0} \eta_f^p,
$$

(105)

$$
U^s_f(K', K_T) = -ik_\beta \Delta \tilde{z}_u u_0^s(K_T)[\tilde{k}_a - \tilde{k}_\beta \cdot \tilde{k}_\beta][\frac{\delta \beta(\tilde{K}_T)}{\beta_0} + \frac{1}{2} \frac{\delta \mu(\tilde{K}_T)}{\mu_0}] \eta_f^s,
$$

(106)

$$
U^{sp}_f(K', K_T) = -ik_\alpha \Delta \tilde{z}[u_0^s(K_T) - \tilde{k}_\beta (u_0^s(K_T) \cdot \tilde{k}_\beta)] \frac{\delta \beta(\tilde{K}_T)}{\beta_0} \eta_f^{sp},
$$

(107)

$$
U^{ss}_f(K', K_T) = -ik_\beta \Delta \tilde{z}[u_0^s(K_T) - \tilde{k}_\beta (u_0^s(K_T) \cdot \tilde{k}_\beta)] \frac{\delta \beta(\tilde{K}_T)}{\beta_0} \eta_f^{ss},
$$

(108)

$$
U^b_p(K_T, z_i) = \frac{i}{2} k_\alpha \tilde{k}_a \Delta \tilde{z} e^{i(k_\alpha + k_\beta) \Delta z/2} \eta_b^p \left[ (-1 + 2 \frac{K_T^2}{k_\alpha^2}) \nu_\rho - V_\alpha - (1 - \frac{2K_T^2}{k_\alpha^2})^2 V_\mu \right],
$$

(109)

$$
U^b_s(K_T, z_i) = \frac{i}{2} k_\beta \Delta \tilde{z} e^{i(k_\alpha + k_\beta) \Delta z/2} \eta_b^s \left[ -1 + \frac{1}{2} \left( \frac{K_T}{k_\alpha} + \frac{K_T}{k_\beta} \right) \right] V_\rho
$$

$$
+ \frac{2}{k_\alpha^2} \left( \frac{K_T}{k_\alpha} + \frac{K_T}{k_\beta} \right)^2 V_\mu - \tilde{k}_\beta \cdot \nu_\beta,
$$

(110)

where $\nu_\rho$ is the distorted P wave amplitude by density perturbation, $V_\rho^p$ is the distorted P wave vector field (displacement) by density perturbation, etc., defined as
\[ V_\rho(K_T) = \int \int d^2x_T e^{-i K_T \cdot x_T} u_0^\rho(x_T, z') \frac{\delta \rho(x_T)}{\rho}, \]
\[ V_\lambda(K_T) = \int \int d^2x_T e^{-i K_T \cdot x_T} u_0^\lambda(x_T, z') \frac{\delta \lambda(x_T)}{\lambda + 2 \mu}, \]
\[ V_\mu(K_T) = \int \int d^2x_T e^{-i K_T \cdot x_T} u_0^\mu(x_T, z') \frac{2 \delta \mu(x_T)}{\lambda + 2 \mu}, \]
\[ V_\rho^p(K_T) = \int \int d^2x_T e^{-i K_T \cdot x_T} u_0^\rho(x_T, z') \frac{\delta \rho(x_T)}{\rho}, \]
\[ V_\mu^p(K_T) = \int \int d^2x_T e^{-i K_T \cdot x_T} u_0^\mu(x_T, z') \frac{\delta \mu(x_T)}{\mu}, \]
\[ V_\rho^S(K_T) = \int \int d^2x_T e^{-i K_T \cdot x_T} u_0^\rho(x_T, z') \frac{\delta \rho(x_T)}{\rho}, \]
\[ V_\mu^S(K_T) = \int \int d^2x_T e^{-i K_T \cdot x_T} u_0^\mu(x_T, z') \frac{\delta \mu(x_T)}{\mu}, \]
\[ \tilde{K}_\beta(K_T) = \left( 2K_T/k_\alpha, 2K_T^2/k_\beta^2 \right), \]

and the factors \( \eta_f^{PP}, \eta_b^{PP}, \eta_f^{PS}, \eta_b^{PS}, \eta_f^{SP}, \eta_b^{SP} \) and \( \eta_f^{SS}, \eta_b^{SS} \) are given by equations (75)-(77). In the above derivation, we assume that the thin-slab is thin enough so that the parameters can be approximated as unchanging along the \( z \) direction, and we also approximate \( \gamma_\alpha \) and \( \gamma_\beta \) in the denominators of equations (100) by \( k_\alpha \) and \( k_\beta \), respectively. Although equations (109)-(112) look a little more complicated in form than those in zero-order method, they can all be implemented using the FFT and the efficiency is not compromised too much. It has been shown that the zero-order complex screen results agree with the thin-slab results only for small incidence angles. As incidence angle increases, the amplitude of the reflected waves deviates gradually from that by the thin-slab method. However, the first order complex screen results are in good agreement with the thin-slab results for fairly wide angles (40° for P wave incidence and 20° for S wave incidence for the French model).

### 6. REFLECTED WAVE FIELD MODELING USING THIN-SLAB METHOD

AVO analysis plays an important role in modern seismic data interpretation in exploration seismology. AVO measures the angle-dependent reflection response of an interface and relates the response to in-situ elastic parameters and fluid contents of the target intervals. For flat interfaces in homogeneous elastic media, AVO curves can be easily calculated theoretically. However, the observed reflection responses of a seismic target are significantly affected by many other factors, such as data collection, data processing and wave propagation effects in heterogeneous media. Before analyzing the AVO responses, these effects should be studied and compensated.

Forward modeling can be useful in understanding wave propagation effects on AVO. In addition, forward modeling can also be useful in interpreting complicated AVO measurements, providing appropriate model parameters for data processing, and developing algorithms of frequency-dependent AVO inversion (Dey-Sarkar and Svatek, 1993). There are several modeling algorithms available for AVO calculation. The reflectivity method is one of the most common methods used for modeling AVO responses in layered media (Fuchs and Müllër, 1971; Simmons and Backus, 1994; Wapenaar et al., 1999). It can generate an exact AVO
response in arbitrarily layered medium, but cannot be used in structures with lateral velocity variations. Ray methods based on various approximations of the Zoeppritz equations are also very common for AVO analysis (Widess, 1973; Simmons and Backus, 1994; Bakke and Ursin, 1998). However, in the presence of thin layers, primaries-only Zoeppritz modeling may produce incorrect results. Another intrinsic limit of the ray methods is its inability in dealing with frequency-dependent scattering associated with heterogeneities. A number of authors applied pseudospectral and finite-difference methods to more complicated geologic models including anelasticity, overburden structure, scattering attenuation, and anisotropy (Chang and McMechan, 1996; Adriansyah and McMechan, 1998; Yoon and McMechan, 1996). In principle, these methods can deal with arbitrarily complicated geologic models. However, they are very time-consuming and memory demanding especially for 3-D models. This is also true for handling structures with thin layers where fine grids must be used.

In this section we apply the elastic thin-slab method to AVO modeling in sedimentary rocks. Several numerical experiments, including reflection coefficient calculations, reflection synthetics with lateral parameter variations, and thin-bed AVO, have been conducted and compared with reflectivity and finite-difference methods. The accuracy and wide-angle capacity of the thin-slab method are demonstrated. Some examples showing the effects of lateral structure variations and random heterogeneities on AVO in sedimentary rocks are presented and analyzed.

6.1. Reflection coefficients of sedimentary interfaces

Reflection coefficients vary as a function of offset from the source (or reflection angle) across an interface. This information is the core of AVO analysis. For either forward modeling or inversion, accurate calculation of reflection coefficients is crucial. Here, we use the thin-slab method to calculate the reflection coefficients at the shale/gas, shale/oil and shale/brine interfaces. A rectangular grid of $1024 \times 200$ is used in the calculation with its parameters given in Table 1. The grid spacings used are 16 m in horizontal direction and 4 m in vertical direction. The interface is located at the depth of 200 m. A taper function is applied to the bottom of the model for eliminating the reflection from the artificial truncation at bottom. Only P-P reflection coefficients are displayed. For reflection coefficient calculation, we take 200 samples (displacement amplitudes in space domain) in the middle of the model for both incident and reflected waves, and define the ratio of their average displacement amplitudes as the reflection coefficient. Figure 12 shows the P-P reflection coefficients for all sets of parameters given in Table 1. The dotted curves represent the corresponding theoretical results. $\phi$ is the percentage porosity, and $\sigma_1$ and $\sigma_2$ are the Poisson's ratios for the shale and sand, respectively. The top panel corresponds to shale/gas interface, the middle panel, shale/oil interface, and the bottom panel, shale/brine interface. We see that the reflection coefficients calculated by the thin-slab method are in excellent agreement with theoretical values for small and medium incident angles ($<30^\circ$) for all cases, and up to a wide angle of incidence ($<40^\circ$) for the 20-percent-porosity gas, oil, and brine sands.

Although the 20-percent-porosity sand has relatively large velocity perturbations, the thin-slab results show better matches to theoretical reflection coefficients for wide angles of incidence than those for 23-percent-porosity and 25-percent-porosity sands at wide angles. It implies that the accuracy and wide-angle capacity of the thin-slab method are related not only to velocity and density perturbations but also to their combinations (the impedance).

The computational efficiency, accuracy, and wide-angle capacity of the thin-slab method are closely related to the amount of perturbations. Generally, the accuracy and wide-angle capacity decrease as perturbations (velocity and/or density) increase. Its computational efficiency also decreases because finer forward steps must be utilized to guarantee the convergence. To see the perturbations of reservoir parameters shown in Table 1, we calculate velocity and density fluctuations relative to the shale (shown in Table 2). The perturbations of the physical parameters used in the models are typical of most sedimentary rocks.

6.2. Reflections from a dipping sandstone reservoir

Figure 13 shows a model of a dipping sandstone reservoir bearing gas, oil, and brine. The dipping angle
of the reservoir is 10° to horizontal plane. The reservoir is thick enough so that the reflections from the top and base are separated in seismograms. Simmons and Backus (1994) used these models to investigate AVO responses associated with angles of incidence, different type interfaces, and porosities of interest. In this section we will use these models to show the accuracy of the thin-slab method for modeling reflections including primary reflections, conversions and diffractions.

Figure 14 displays the synthetic seismograms with incident plane waves for three different porosities (labeled in each panel). The plane P-wave is vertically downward incident to the model. The left column corresponds to the finite-difference results and the right panel, to the thin-slab results. The last two arrivals are the converted shear waves produced at the top and base of the reservoir and propagated in the shale. Single-leg (one converted wave path in the layer) and double-leg (two converted wave paths in the layer) converted shear waves are weak at a small angle of incidence and overlap with the primary reflections from the base. Note that in the thin-slab results (the right column in Figure 14) the multiples within the sandstone are neglected. Comparing the two columns, we see that the thin-slab results are in good agreement with the finite-difference results, implying that the multiples are not significant in sedimentary rocks.

6.3. Reservoir reflections with scattering and attenuation from a heterogeneous overburden

We examine the effect of scattering and attenuation associated with heterogeneities in sedimentary rocks on AVO using the thin-slab modeling method. First we study the effects of random scattering. The reservoir is a flat model generated by rotating the model in Figure 13. A 2-D random field with exponential correlation function is used to perturb the velocity and density parameters of the sedimentary rocks (Figure 15). The correlation lengths of the random field are 100 m in horizontal direction and 40 m in depth. The rms values used are 1%, 2% and 3%. Note that both P- and S-wave velocities have the same distributions and rms perturbations; while the density has the same distribution but only one half of the rms value of velocity perturbation. Only velocity rms values are indicated in the text and figures. For simplicity, we consider a plane P-wave vertically incident on the reservoir. Figure 15 shows the model and the snapshots at t = 0.2 s, 0.4 s and 0.6 s, respectively. The porosity of the sand is 25% and rms velocity perturbation is 2%. In Figure 15 we see that abundant coda waves are produced. The wavefronts of both forward and reflected waves are distorted. For the reflected waves from shale/oil and shale/brine interfaces, serious distortions can be seen. Figure 16 shows the maximum magnitudes of the responses from the top interface of the sand. The vertical axis is the logarithmic amplitude, and (a), (b) and (c) correspond to porosities of 20%, 23%, and 25%, respectively. Solid lines correspond to calculations in models without overburden heterogeneities, in which the fluctuations in amplitudes are caused by the interference of boundary diffractions. The dotted, dashed and dotted-dashed lines correspond to overburden models having rms random velocity fluctuations of 1%, 2% and 3%, respectively. The vertical interfaces separating gas, oil and brine are indicated by arrows. Note that for an overlying shale with heterogeneities as small as rms=1%, the piece-wise uniform reflection amplitudes become fluctuating due to the focusing and defocusing of waves. As the velocity fluctuation increases, stronger amplitude fluctuations are generated. The amplitude fluctuations are closely related to the statistical properties of heterogeneities. The existence of a laterally varying overburden layer has serious effects on reflecting waves from reservoirs. It generates scattered waves and affects the reflection characteristics of local interfaces. For weak reflection sands, the scattering effect from heterogeneous overburden could be important and must be taken into account for AVO analysis. The top panel shows reservoir model bearing gas, oil and brine, respectively. The formation is anelastic and heterogeneous. The lower three panels show AVO’s for various kinds of interfaces: (a) shale/gas, (b) shale/oil, and (c) shale/brine. For each kind of interface, three different constant Q’s (Q = infinity, 150, 50) are given to shale. The sand has constant Qp = Qs = 10. The correlation lengths of the random field for perturbing Q and elastic parameters are the same. The rms values are 4% for elastic parameters and 25% for Q.

Next, we show combined effects of random scattering and intrinsic attenuation. In practice, the geologic models may contain arbitrary spatial variations in compressional- and shear-wave quality factors, as well as density and velocities (see the top panel in Figure 17). For each kind of interface, three different averaged Q’s (i.e., Q=50, 150, infinity) are given to shale, and the sands have quality factors of Qp = Qs = 10. The
correlation lengths of the random field for perturbing Q and elastic parameters are the same. The rms values are 4% for elastic parameters and 25% for Q. The source and receiver array are located in shale and 1200 m away from the interface. The dotted lines in Figure 17 correspond to the homogeneous cases with constant Q’s. We see that intrinsic attenuation mainly affect the absolute reflected amplitudes and heterogeneities in both Q and elastic parameters affect local amplitude fluctuation with offset. In summary, the AVO responses of the target subsurface have been significantly deformed due to both the heterogeneities and intrinsic attenuation.

6.4. Thin layer AVO response

The amplitude response of a thin-bed has drawn increasing interest in hydrocarbon interpretation because large quantities of gas reserves are found to be trapped with thin sands. The AVO response of a thin-bed is different from that of a thick bed because of the effects of wave interference, conversion and tunneling. For a thin-bed with thickness much less than the predominant wavelength, the grid methods such as finite-difference and finite-element methods are too costly for modeling reflections in AVO analysis. We investigate the ability of the thin-slab method for handling elastic thin-bed reflections. The thin-bed used is 5 m thick, located at a depth of 1500 m and is filled with an oil sand of porosity 20%. The predominant wavelength is 125 m, being 25 times greater than the thickness of the thin-bed. The surrounding medium is shale. Source and receivers are on the top of the model. For the calculation using the thin-slab method, the grid spacing in the horizontal direction is 10 m and in the vertical direction is 10 m for the background area and 0.5 m for the thin-bed.

Figure 18 displays reflection seismograms calculated by the reflectivity method (a) and thin-slab method (b). NMO corrections are applied to the top of the sand and reduced time is used in showing seismograms. In Figure 18, A represents reflections from the top of the model, B gives reflections from both the top and base but no converted waves are involved, C includes all waves (exact solution) and D gives reflections from both the top and base including both single-leg and double-leg converted shear waves. Comparing A and B, we see the big difference between AVOs of a single impedance interface and two closely located impedance interfaces. B is equivalent to primary-only reflections. Comparing B with C and D, we see the important effect of locally converted shear waves that can alter amplitude variation with offset. Comparing C and D, only small differences exist at large offsets. In this case, the effect of multiples is negligible.

For a homogeneous thin-slab under small-angle approximation, equation for one-return reflection can be simplified into (Wu, 1996; Xie and Wu, 2001)

$$U^p(K_T) = -i k_u \Delta z \frac{\delta Z_\alpha}{Z_\alpha} u^0_{\alpha}(K_T),$$

where $Z_\alpha$ is the P-wave impedance. The above equation presents not only the amplitude of the thin-bed reflection but also the change in wavelet. Therefore, the thin-slab propagators represent the response of an elastic heterogeneous thin-layer. Since the choice of $\Delta z$ or step interval in wavefield extrapolation is flexible and can vary according to local heterogeneities, the thin-slab method can naturally handle arbitrarily thin layers.

6.5. AVO response in laterally varying media

In this section we use a simple model containing both a truncated salt layer and a thin gas sand, which is located 200 m below the salt layer (Figure 19a), to examine the AVO response using the thin-slab method.

The parameters for the gas sand are given in Table 1 and the corresponding porosity is 20%. The parameters for salt are $V_p = 4.48 km/s$, $V_s = 2.594 km/s$, $\rho = 2.1 g/cm^3$. The thicknesses of the salt layers are 20 m, 40 m and 80 m, respectively. The grid spacing is the same as that used in Figure 18 except for inside the salt body, where a 0.1 m grid space is used. In Figure 19, the maximum negative amplitudes of the thin-bed responses are picked and plotted versus offset. The solid curve represents the thin-bed (without salt) AVO. The AVO trend of a thin-bed is different from that shown in Figure 12 (shale/gas, $\phi = 20$) where the reflection coefficient increases with incident angles up to 45° or an offset of 3 km. The dotted, dashed and dotted-dashed curves correspond to salt layer thicknesses of 20 m, 40 m, and 80 m, respectively. We see that
AVOs are altered due to the presence of the salt layer. The salt layer can cause diffraction, defocusing, conversion and transmission loss. The diffraction affects AVO by causing interference between diffracted waves from the left side of the salt layer and reflected waves from the thin-bed at receiver locations. This interference causes local variations in AVO measurements. The transmission loss affects all the reflections passing through the salt body and causes a systematic decrease in AVO.

Figure 20 shows the combined effects of a truncated salt layer and random heterogeneities on AVO. Heterogeneities are introduced into the model shown in Figure 19a. The correlation lengths of the 2D random field are 100 m in horizontal direction and 40 m in vertical direction. The rms fluctuations used are 1%, 2%, and 3%, respectively. The four panels in Figure 20 correspond to the cases of zero-salt, 20m, 40m and 80m-thick salt (all labeled in the figure). The case of zero-salt shows the effect from heterogeneities alone. The focusing and defocusing of the spatially correlated heterogeneities produce the local variation in reflected amplitudes versus offset which becomes significant for sedimentary models with weak reflections. Interpretation of AVO observations based on homogeneous elastic models will therefore bias from actual properties of the target. The frequency and amplitude variations of reflections are closely related to the source spectrum and the statistical properties of velocity perturbations, although the overall AVO trends are still controlled by the target properties and overburden structures. To improve AVO analysis in sedimentary rocks, it is necessary to have sufficient acquisition apertures and take into account of the overburden structures including random heterogeneities and thin-bed effects.

7. CONCLUSIONS

In this chapter renormalized MFSB (multiple-forescattering single-backscattering) equations and the dual-domain expressions for scalar, acoustic and elastic waves are treated in a unified approach. The De Wolf approximation neglects the reverberations (internal multiples) inside, but can model all the forward scattering phenomena, such as focusing/defocusing, diffraction, refraction, interference, etc., and the primary reflections. The De Wolf approximation can be considered as the first order term in a De Wolf multiple scattering series. This single backscattering signal defined this way is the primary reflections from the real medium, and is different from the Born approximation where the incident field and the Green’s function are both defined in the reference medium (a homogeneous medium). It is also different from the first term of the generalized Bremmer series, which uses a high-frequency asymptotic solution (a WKBJ-like solution) as the Green’s function (in an equivalent reference medium).

Two versions of the one-return method (using MFSB approximation) are given: One is the wide-angle dual-domain thin-slab approximation; the other is the screen approximation. The latter involves a small-angle approximation for the wave-medium interaction. Q-factor from intrinsic attenuations has been incorporated into the algorithm by the introduction of complex velocities. The reflectivity method has also been incorporated to the thin-slab modeling to increase the efficiency.

The theory and methods are applied to the fast calculation of synthetic seismograms. For weak heterogeneities (±15% perturbation), good agreement between the one-return method and finite difference simulations verifies the validity of the one-return approach. However, the one-return approach is about 2-3 orders of magnitude faster than the elastic FD algorithm. The method can be applied to the fast modeling of AVA responses for a complex reservoir with heterogeneous overburdens. The method can be applied to most of the real cases where the perturbations of P- and S-wave velocities are around or smaller than 30%. The influences of heterogeneous or random overburdens, thin-bedding, Q-variation, irregular salt layer, etc., can be studied using this modeling technique.

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Helpful discussions with and comments from M. Fehler, L.J. Huang, R. Hobbs, S. Jin, C. Mosher, and K. Wapenaar are greatly appreciated. This work was supported by the Department of Energy through various contracts and by the WTOPI (Wavelet Transform on Propagation and Imaging for seismic exploration) Research Consortium at the University of California, Santa Cruz. Facility support from the W.M. Keck Foundation is also acknowledged. Contribution number 488 of CSIDE, IGPP, University of California, Santa
Cruz.

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Table 1. Reservoir model.

<table>
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<tr>
<td>brine</td>
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Table 2. Perturbations of reservoir parameter.

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<td>$%$</td>
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<td>oil</td>
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Figure 1: Sketch showing the meaning of the De Wolf (one-return) approximation.

Figure 2: Schematic illustration of the thin-slab method for implementing the one-return approximation: (a) Original medium is sliced into thin-slabs, (b) iterative procedure of transmitted and reflected wave calculations by the one-return approximation.
Figure 3: Geometry for the derivation of the thin-slab method.

Figure 4: Comparisons of reflection coefficients calculated by the thin-slab method and the exact solutions. The upper panel corresponds to P wave and the lower panel to S wave incidences.
respectively. The bottom layer of the model has 10% velocity perturbations for both P and S waves with respect to the top layer.

![Graphs comparing reflection coefficients](image)

Figure 5: Comparisons of reflection coefficients calculated by the thin-slab method and the exact solutions. The perturbation of the bottom layer is –10% with respect to the top layer for both P and S wave velocities.
Figure 6: Comparisons of reflection coefficients calculated by the thin-slab method and the exact solutions. The perturbation of the bottom layer is 20% with respect to the top layer for both P and S wave velocities.

Figure 7: Comparisons of reflection coefficients calculated by the thin-slab method and the exact solutions. The perturbation of the bottom layer is –20% with respect to the top layer for both P and S wave velocities.

Figure 8: A 2-D heterogeneous model (French, 1974) with irregular interface used to test the validity and accuracy of the thin-slab method. The background medium has $\alpha_0 = 3.6 \text{ km/s}$, $\beta_0 = 2.08 \text{ km/s}$ and $\rho_0 = 2.2 \text{ g/cm}^3$. The layer in black color has a perturbation of –20% for both P and S wave velocities.
Figure 9: Comparison of synthetic seismograms calculated by finite difference (solid) and by thin-slab methods (dashed). Curves in (a) and (b) are horizontal and vertical components of displacement, respectively.
Figure 10: The 3D French model for the numerical tests. The fat solid lines delineate the horizontal surface at which the thin-slab method is connected to the reflectivity method. The background medium has the same parameters as those in Figure 8. The structure in grey color has a perturbation of -10% for both P and S velocities.
Figure 11: Comparison of synthetic seismograms calculated by finite difference (solid lines) and by thin-slab methods (dashed lines). From top to the bottom are X-, Y- and Z-components of displacement, respectively.
Figure 12: P-P reflection coefficients at different type of interfaces: shale/gas (top), shale/oil (middle), and shale/brine (bottom). All formation parameters are listed in Table 1. The solid curves are calculated by the thin-slab method and the dotted lines, by the Zoeppritz equations.
Figure 13: Model of a dipping sandstone reservoir filled with gas, oil, and brine. The reservoir is thick enough so that the reflections from the top and base can be separated.
Figure 14: Comparisons of plane wave seismograms calculated by finite-difference (left column) and thin-slab (right column) methods for a dipping reservoir model shown in Figure 13.
Figure 15: (a) Sandstone model with heterogeneities, (b) to (d) are Snapshots at $t = 0.2$ s, $0.4$ s, and $0.6$ s, respectively. A $30$ Hz plane P-wave source is vertically incident from the top of the model.
Figure 16: Effect of scattering by heterogeneities on reflected amplitudes of reservoirs shown in Figure 15.
Figure 17: The top panel shows reservoir model bearing gas, oil and brine, respectively. The formation is anelastic and heterogeneous. The lower three panels show AVO’s for various kinds of interfaces: (a) shale/gas, (b) shale/oil, and (c) shale/brine. For each kind of interface, three different constant Q’s (Q = infinity, 150, 50) are given to shale. The sand has constant $Q_P = Q_S = 10$. The correlation lengths of the random field for perturbing Q and elastic parameters are the same. The rms values are 4% for elastic parameters and 25% for Q.
Figure 18: Reflection seismograms for an oil sand model. The oil sand is 5 m thick and at depth of 1500 m. (a) By reflectivity method, (b) by thin-slab method. In the figure, A represents reflections from the top of the model (thick layer), B, reflections from both top and base but no converted waves included, C, all waves included (exact solution), and D, reflections from both top and base including single-leg and double-leg converted shear waves.
Figure 19: (a) Model containing a truncated salt layer with thickness $d$ and a thin gas sand layer below the salt, (b) Angle-dependent reflections (AVO) of a thin gas layer with reflections from the salt layer of different thickness.
Figure 20: AVO’s of the thin gas layer with the combined effects of lateral structure variation and random heterogeneities.
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1995


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